Master Thesis

Ice Flow Simulations of the Mocho-Choshuenco Ice Cap

Matthias Scheiter
Matriculation Number: 57195
M.Sc. Geophysics

First Supervisor: Prof. Dr. Klaus Spitzer
Institut für Geophysik und Geoinformatik
TU Bergakademie Freiberg

Second Supervisor: Dr. Marius Schaefer
Instituto de Ciencias Físicas y Matemáticas
Universidad Austral de Chile

March 2019
(Last Revision: April 30, 2019)
Abstract

As most glaciers in southern Chile, the Mocho-Choshuenco ice cap has experienced large mass losses during the last decades. The projected global warming during the 21st century gives reason to expect a further decreasing surface mass balance. In order to estimate the consequences exerted by temperature enhancement on the dynamic behaviour of the ice cap, the ice sheet model SICOPOLIS is applied to the ice cap. To date, the Mocho-Choshuenco ice cap is by far the smallest cryospheric object analysed with SICOPOLIS. First, the equations of the shallow ice approximation which are incorporated by the model are derived from the basic principles of mass and momentum balance. Then, a new surface mass balance parametrisation accounting for local differences in the equilibrium line altitude due to wind deposition and solar radiation effects is introduced and tested. Afterwards, the relevance of sliding on ice flow of the ice cap is analysed. Finally, the ice cap is modelled up to the year 2065 under the increasing temperatures of the RCP2.6 and RCP8.5 scenarios. The shallow ice approximation proves to work well on the plateau of the ice cap, but shows difficulties at locations with higher slopes. The new surface mass balance parametrisation is able to improve the simulation of the ice extent, while sliding improves the ice thickness and velocity reproduction of the observed ice cap. However, the total ice volume is higher in the model than in reality. The future scenarios predict a strong decrease in ice mass: 47% and 74% of the ice cap would disappear until 2065 under the RCP2.6 and RCP8.5 scenarios, respectively. This suggests a possible complete disappearance of the ice cap until the end of the century under the most pessimistic scenario.
## Contents

1. Introduction 5

2. Theoretical Background 7
   2.1. Ice Thickness Equation (Conservation of Mass) 7
   2.2. Full Stokes Equations (Conservation of Momentum) 9
   2.3. Glen’s Flow Law 10
   2.4. Weertman’s Sliding Law 12
   2.5. Approximations to the Full Stokes Equations 13
      2.5.1. Hydrostatic Approximation 13
      2.5.2. First Order Approximation 14
      2.5.3. Shallow Ice Approximation 14

3. Methods 20
   3.1. Sigma Transformation 20
   3.2. Discretisation 21
   3.3. One-step Methods 23
   3.4. Numerical Solution of the Ice Thickness Equation 26
      3.4.1. Horizontal Velocity 26
      3.4.2. Vertical Velocity Integration 27
      3.4.3. Ice Thickness Equation 27

4. Study Object 28
   4.1. The Mocho-Choshuenco Volcanic Complex 28
   4.2. Input Data from Glaciological Studies on the Ice Cap 29

5. Results 32
   5.1. Aspect-dependent SMB Parametrisation 32
      5.1.1. Derivation of the Parametrisation 32
      5.1.2. Influence of the Parametrisation 34
      5.1.3. Steady-State Ice Volume 38
   5.2. Relevance of Sliding 39
      5.2.1. Ice Thickness Reduction 39
      5.2.2. Surface Velocity Contrasts 41
      5.2.3. Ice Volume Contrasts 43
      5.2.4. Variation along a Flowline 44
   5.3. Influence of Resolution 47
      5.3.1. Spatial Resolution Variation 47
1. Introduction

Over the past few decades, glaciers have become a symbol of global change, contributing to societal awareness about its impact on the sensible and complex planet Earth system. They are present on all continents, and the overall ice mass outside the ice sheets of Greenland and Antarctica cover a total area of more than 700 km$^2$, distributed over around 215,000 glaciers and ice caps. Since the mid-19th century, glacier mass loss has increased continuously, and in the first years of the current century, it has doubled as compared to the previous 50 years (Zemp et al., 2009).

In the southern hemisphere, the great majority of ice outside Antarctica is concentrated in South America (Pfeffer et al., 2014). Here, the observed mass balance in almost all sub-regions is clearly negative, and 83% of the melt originate from Patagonia with its huge Northern and Southern Patagonian Icefields (Braun et al., 2019). More than half of the Patagonian glaciers analysed by Sakakibara and Sugiyama (2014) experienced significant retreat since the 1980s. Apart from a major impact of ice dynamics on glacier behaviour (Mouginot and Rignot, 2015), Patagonia is characterized by high precipitation rates and temperatures. This leads to an exceptionally high mass turnover (Schaefer et al. (2013), Schaefer et al. (2015)) which is present until the northernmost boundary of Patagonia (Schaefer et al., 2017).

The drastic observed changes in ice masses and the projected temperature increment throughout the 21st century emphasize the need to further study the behaviour of the cryosphere. To this end, ice sheet models have been developed which are based on the theory of continuum mechanics and numerical methods (Greve and Blatter, 2009) and are found in a wide spectrum of complexities (Kirchner et al., 2011). The most sophisticated ice sheet models solve the full system of Stokes equations, one example is the model Elmer/Ice (Gagliardini et al., 2013). On the other side of the spectrum, the shallow ice approximation (SIA) is found. It makes use of several characteristics of ice sheet geometry to simplify the system of equations that needs to be solved significantly (Greve and Blatter, 2009). The SIA is implemented in the ice sheet model SICOPOLIS (Greve, 1997) which has been applied to a wide range of study objects, especially the Greenland ice sheet (e.g. Calov et al. (2018)).

Recently, SICOPOLIS was applied to the Mocho-Choshuenco ice cap in southern Chile (Flandez, 2017). With an area of only around 17 km$^2$ (Rivera et al., 2005), this ice cap is very small compared to the objects usually modelled with SICOPOLIS. One main finding was a high dependence of the ice cap on climatic factors, as the ice-covered area responded considerably to changes in equilibrium line altitude (ELA). However, the sim-
ulated ice extent deviated locally with respect to the observed ice margins, leading to the conclusion that local surface mass balance (SMB) patterns are more complex than assumed by the model.

The above mentioned great abundance of extra-polar glaciers and ice caps, especially in South America, and their high amounts of stored freshwater make them valuable study objects. The present thesis will intensify the application of SICOPOLIS to a small ice cap in order to prove the ability of the model to reproduce its current behaviour. For this purpose, the current SMB parametrisation will be further developed and the influence of ice dynamics on the ice cap behaviour will be analysed. A further objective is to predict the fate of the ice cap in response to the projected 21st century temperature increment.

This thesis starts in Chapter 2 with a derivation of the most important equations used in SICOPOLIS. Chapter 3 gives an outline of the numerical methods that are used to solve the system of equations. In Chapter 4, the Mocho-Choshuenco ice cap is introduced together with the glaciological studies that have been performed on it so far. Chapter 5 presents and analyses the results of the simulations. Chapter 6 discusses the results, makes suggestions for future simulations and gives an outlook.
2. Theoretical Background

In the following sections, the theory behind the most important equations used in ice flow modelling is derived and explained. These equations are also used in SICOPOLIS and thus provide a basis for understanding and interpreting outcomes of the model. Most equations and derivations follow Greve and Blatter (2009) closely. Appendix A gives a short introduction on the stress tensor \( T \) and the strain rate tensor \( T^D \) which are used in the following.

2.1. Ice Thickness Equation (Conservation of Mass)

First, the ice thickness equation will be derived which will facilitate the calculation of ice thickness on a two-dimensional grid for a specific moment of time. The starting point for this derivation is the continuity equation

\[
\rho \nabla \cdot \mathbf{v} = \frac{\partial \rho}{\partial t},
\]

where \( \mathbf{v} \) is the velocity and \( \rho \) is mass density. It describes the conservation of mass: A temporal change of mass density \( \frac{\partial \rho}{\partial t} \) inside a volume element is only possible when a volume flux \( \nabla \cdot \mathbf{v} \) from or to the volume element happens. In other words, there are no sources or sinks of mass.

Assuming incompressibility of ice \( (\frac{\partial \rho}{\partial t} = 0) \) yields the incompressible continuity equation. In order to derive the ice thickness equation, it will be transformed in the following:

\[
\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0
\]

\[
\int_b^h \frac{\partial v_x}{\partial x} \, dz + \int_b^h \frac{\partial v_y}{\partial y} \, dz + \int_b^h \frac{\partial v_z}{\partial z} \, dz = 0 \tag{2.1}
\]

The Leibniz rule

\[
\frac{\partial}{\partial x} \int_b^h v_x \, dz = \int_b^h \frac{\partial v_x}{\partial x} \, dz + v_x \bigg|_{z=b}^{z=h} \frac{\partial b}{\partial x} - v_x \bigg|_{z=b}^{z=h} \frac{\partial h}{\partial x}
\]

can be rewritten to yield an expression for the first term in Equation 2.1:
\[ \int_b^h \frac{\partial v_x}{\partial x} \, dz = \frac{\partial}{\partial x} \int_b^h v_x \, dz - v_x \bigg|_{z=h} \frac{\partial h}{\partial x} + v_x \bigg|_{z=b} \frac{\partial b}{\partial x} \]

The second term can be expressed analogously,

\[ \int_b^h \frac{\partial v_y}{\partial y} \, dz = \frac{\partial}{\partial y} \int_b^h v_y \, dz - v_y \bigg|_{z=h} \frac{\partial h}{\partial y} + v_y \bigg|_{z=b} \frac{\partial b}{\partial y}, \]

and the third term gives

\[ \int_b^h \frac{\partial v_z}{\partial z} \, dz = v_z \bigg|_{z=h} - v_z \bigg|_{z=b}. \]

Replacing the respective terms in Equation 2.1 yields

\[ \frac{\partial}{\partial x} \int_b^h v_x \, dz + \frac{\partial}{\partial y} \int_b^h v_y \, dz - v_x \bigg|_{z=h} \frac{\partial h}{\partial x} - v_y \bigg|_{z=h} \frac{\partial h}{\partial y} + v_z \bigg|_{z=b} \frac{\partial b}{\partial x} + v_y \bigg|_{z=b} \frac{\partial h}{\partial y} - v_z \bigg|_{z=b} = 0. \quad (2.2) \]

The kinematic boundary conditions\(^1\) are

\[ \frac{\partial h}{\partial t} = a_s - v_x \frac{\partial h}{\partial x} - v_y \frac{\partial h}{\partial y} + v_z \quad \text{for the surface, and} \]

\[ \frac{\partial b}{\partial t} = a_b - v_x \frac{\partial b}{\partial x} - v_y \frac{\partial h}{\partial y} + v_z \quad \text{for the base.} \]

In these equations, ice surface height \( h \) (or ice base height \( b \)) changes with time if one of the terms on the right side of the equation is greater or less than zero: \( a_s \) (\( a_b \)) is surface (basal) mass balance, \( v_x \frac{\partial h}{\partial x} \) and \( v_y \frac{\partial h}{\partial y} \) (\( v_x \frac{\partial b}{\partial x} \) and \( v_y \frac{\partial b}{\partial y} \)) are topography advection terms and \( v_z \) is the vertical velocity.

Since several terms in the kinematic boundary conditions also appear in Equation 2.2, they can be replaced leading to

\[ \frac{\partial}{\partial x} \int_b^h v_x \, dz + \frac{\partial}{\partial y} \int_b^h v_y \, dz + \frac{\partial h}{\partial t} - a_s - \frac{\partial b}{\partial t} + a_b = 0. \]

After defining the volume flux \( q \) as the vertically averaged velocity,

\(^1\)Greve and Blatter (2009), Equations 5.21 and 5.31
\[
q = \begin{pmatrix} q_x \\ q_y \end{pmatrix} = H \begin{pmatrix} v_x \\ v_y \end{pmatrix}
\]
with \( v_{z/y} = \frac{1}{H} \int_{b}^{h} v_{z/y} \, dz \), (2.3)

we obtain the equation

\[
\frac{\partial h}{\partial t} - \frac{\partial b}{\partial t} = \nabla \cdot q + a_s - a_b,
\]

and inserting the ice thickness \( H = h - b \) finally yields the ice thickness equation:

\[
\frac{\partial H}{\partial t} = -\nabla \cdot q + a_s - a_b \tag{2.4}
\]

It can also be understood as a depth averaged continuity equation in which ice flow only depends on the horizontal coordinates \( x \) and \( y \), but not on the vertical coordinate \( z \).

### 2.2. Full Stokes Equations (Conservation of Momentum)

The momentum balance equation reads

\[
\rho \frac{dv}{dt} = \nabla \cdot T + f, \tag{2.5}
\]

where \( T \) is Cauchy’s stress tensor (see Appendix A.1) and \( f \) is the volume force which is here the effective force of gravity:

\[
f = \rho g
\]

\( \rho \approx 910 \frac{kg}{m^3} \) can be estimated to be constant and \( g = -ge_z \), so it is directed downwards with a value of \( g = 9.81 \frac{m}{s^2} \).

\( \nabla \cdot T \) can be expressed in another way:

\[
\nabla \cdot T = \nabla \cdot (T^D + p) = \nabla \cdot T^D + \nabla p
\]

In this way, the pressure gradient \( \nabla p \) appears implicitly in Equation 2.5 and can be compared to the acceleration term \( \rho \frac{dv}{dt} \). To do so, some typical scale values of ice sheets are necessary:
horizontal extension: \( L \approx 1000 \text{ km} \)
vertical extension: \( H \approx 1 \text{ km} \)
horizontal velocity: \( U \approx 100 \text{ m} \text{ s}^{-1} \)
vertical velocity: \( W \approx 0.1 \text{ m} \text{ s}^{-1} \)
pressure: \( P = \rho g H \approx 10 \text{ MPa} \)
time: \( \frac{L}{U} = \frac{H}{W} \approx 10000 \text{ a} \)

Relating the acceleration term \( \rho \frac{U}{L} \) and pressure gradient \( \frac{P}{L} \) gives the Froude number

\[
Fr = \frac{\rho \frac{U}{L}}{\rho g} = \frac{\rho \frac{U^2}{gH}}{g} \\
\approx \frac{(100 \text{ m})^2}{10 \text{ m}^2 \cdot 1000 \text{ m}} \approx \frac{10^4 \text{ m}^2 \text{ s}^{-2}}{10^{15} \text{ s}^2 \cdot 10^4 \text{ m}^2} \\
= 10^{-15}
\]

so that \( \frac{\text{d}u}{\text{d}t} \ll \nabla p \) and the acceleration term in the momentum balance can be neglected:

\[
-f = \nabla \cdot T
\]

\[
\rho g = \begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{pmatrix} \begin{pmatrix}
T_{xx} & T_{xy} & T_{xz} \\
T_{yx} & T_{yy} & T_{yz} \\
T_{zx} & T_{zy} & T_{zz}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
0 \\
\rho g
\end{pmatrix} = \begin{pmatrix}
\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \\
\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} \\
\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z}
\end{pmatrix}
\]

Written as three equations, this is the system of the full Stokes equations:

\[
\begin{align*}
\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} &= 0 \\
\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} &= 0 \\
\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} &= \rho g
\end{align*}
\]

### 2.3. Glen’s Flow Law

In glaciological studies, the most commonly used constitutive equation for ice flow is Glen’s flow law (Glen, 1955),
\[ D = \frac{1}{2\eta(T',\sigma_e)} T^D, \]  

which relates the deviatoric stress tensor \( T^D \) to the strain rate tensor \( D \) via the viscosity \( \eta \). In other words, it calculates deformation \( (D) \) depending on material properties \( (\eta) \) and the forces acting in the body \( (T^D) \).

Viscosity itself is calculated by

\[ \eta(T',\sigma_e) = \frac{1}{2A(T')\sigma_e^{n-1}} \]

and depends on two quantities, temperature relative to the melting point \( T' \) through the rate factor \( A(T') \) and effective stress \( \sigma_e \) with the exponent \( n \). The latter is the square root of the second invariant of the deviatoric stress tensor \( T^D \):

\[ \sigma_e = \sqrt{II_{T^D}} = \sqrt{\frac{1}{2} \text{tr}(T^D)^2} \]

\[ = \sqrt{\frac{1}{2} \left((T^D_{xx})^2 + (T^D_{yy})^2 + (T^D_{zz})^2\right) + T^2_{xy} + T^2_{xz} + T^2_{yz}} \]

\( \sigma_e \) is independent of the chosen coordinate system and indicates the deviation of stresses from the hydrostatic state. The stress exponent \( n \) is usually assumed to be 3, based on experimental analyses.

The rate factor \( A(T') \) follows the Arrhenius relationship,

\[ A(T') = A_0 \cdot e^{-\frac{Q}{RT'}} \]

with the pre-exponential constant \( A_0 \), activation energy \( Q \) and the universal gas constant \( R = 8.314 \frac{J}{\text{mol} \cdot \text{K}} \).

\( T' \), the temperature relative to the melting point is defined by the equation

\[ T' = T_{loc} + \beta p, \]

with local temperature \( T_{loc} \), pressure \( p \) and the Clausius-Clapeyron constant \( \beta \) which describes the influence of pressure changes on the melting point. For glacier ice, a value of \( \beta = 9.8 \cdot 10^{-8} \frac{\text{K}}{\text{Pa}} \) was found which results in a decrease of the melting point by 0.87 K per kilometre of overburden ice. Therefore, \( \beta p \) can be interpreted as a correction summand for pressure and ice starts to melt when \( T' > 273.15 \text{K} \).
The Mocho-Choshuenco ice cap, however, is assumed to consist of temperate ice due to the high mean atmospheric temperatures (Schaefer et al., 2017). Therefore, temperature $T$ is always at the pressure melting point, leading to a constant $T' = 273.15$ K and thus a constant $A(T')$ with the following constant values for the given temperature:

\[
A_0 = 1.916 \cdot 10^3 \cdot \frac{1}{\text{sPa}^3}
\]

\[
Q = 139 \text{ kJ mol}^{-1}
\]

Since temperature relative to the melting point $T'$ is assumed to be constant, the temperature equation or energy balance equation loses its relevance for application at Mocho-Choshuenco ice cap. For polythermal glaciers and ice sheets, however, it is an important part of the set of equations and needs to be included into the model.

### 2.4. Weertman’s Sliding Law

In order to estimate the sliding behaviour of a glacier at the ice-base interface, Weertman’s sliding law (Weertman, 1957) provides a frequently used parametrisation. It relates basal velocity $v_b$, basal shear stress $\tau_b$ and basal normal pressure $p_b$ in the following way:

\[
v_b = -C_b \frac{\tau_b}{p_b^p} \mathbf{e}_t \tag{2.8}
\]

Here, $\mathbf{e}_t$ is the unit vector in the direction of flow and $C_b$ is a sliding coefficient which can be used as a calibration parameter for the ice flow model. This relationship only holds for a temperate basis where temperature is at the pressure melting point. For cold ice, sliding velocity is assumed to be zero. The basal sliding exponents $p$ and $q$ usually get value pairs such as $(p, q) = (1, 0)$ or $(p, q) = (3, 2)$.

An alternative way to define basal sliding is via the effective pressure, $p_{\text{eff}} = p_b - p_w$, where $p_w$ is basal water pressure:

\[
v_b = C_2 \frac{\tau_b}{p_{\text{eff}}^q} \mathbf{e}_t
\]

Thus, higher water pressure reduces the effective pressure and consequently leads to a higher sliding velocity $v_b$. This behaviour is consistent with field observations.

At places where the glacier is not resting on a hard rock bed but on sediment, basal velocity is enhanced by deformation of the sediment layer. Assuming simple shear in a sediment with deformation behaviour of a Newtonian fluid, the sliding law can be expressed as
\[ v_b = \frac{d}{\eta_{sed}} \tau_b e_t, \]

where \( \eta_{sed} \) is the viscosity of the sediment and \( d \) denotes its thickness which is assumed to be very small.

### 2.5. Approximations to the Full Stokes Equations

In the following, approximations to the full Stokes equations are made which simplify the solution of the problem significantly. They are based on observations made on the behaviour of ice sheets and their characteristic shape: the horizontal dimension of ice sheets is typically much greater than their vertical dimension. Together with the nearly horizontal surface and base slopes, an ice flow behaviour similar to simple shear deformation can be assumed.

#### 2.5.1. Hydrostatic Approximation

In the hydrostatic approximation (Section 5.2 in Greve and Blatter (2009)), first simplifications of the full Stokes equations 2.6 are made: the stress tensor components \( T_{zx} \) and \( T_{zy} \) are neglected because these shear stresses are much smaller than the vertical normal stress (assuming a typical viscosity of ice of \( 10^{14} \text{ Pa s} \)):

\[ T_{zx} = \eta \frac{\partial v_x}{\partial z} = \eta \frac{U}{H} \approx 300 \text{ kPa} \]
\[ T_{zz} \approx P \approx 10 \text{ MPa} \]

Thus, the full system of Stokes equations reduces to:

\[
\begin{align*}
\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} &= 0 \\
\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} &= 0 \\
\frac{\partial T_{zz}}{\partial z} &= \rho g
\end{align*}
\]

Integrating the third equation gives

\[ T_{zz} = -\rho g (h - z), \tag{2.9} \]

so that the vertical normal component of the stress tensor equals hydrostatic pressure, leading to the name of the hydrostatic approximation. Based on this result, two equations can be derived which together with Glen’s flow law and the ice thickness equation
compose a system of four equations to solve for the four unknowns $v_x, v_y, v_z$ and $H$ (Greve and Blatter, 2009). In the following, further approximations will be made leading finally to the shallow ice approximation that is incorporated in SICOPOLIS.

2.5.2. First Order Approximation

In the first order approximation (Section 5.3 in Greve and Blatter (2009)), not the Stokes equations, but the formulation of the deformation tensor $D$ change. This simplification is based on the observation that in an ice sheet, the vertical variation of the horizontal velocity components ($\frac{\partial v_x}{\partial z}$, $\frac{\partial v_y}{\partial z}$) is much bigger than the horizontal variation of the vertical velocity components ($\frac{\partial v_x}{\partial y}$, $\frac{\partial v_y}{\partial x}$). The latter are therefore neglected and the stress tensor $D$ (cf. Appendix A.2) changes to

$$D = \frac{1}{2} \begin{pmatrix}
2\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} & \frac{\partial v_x}{\partial z} + \frac{\partial v_y}{\partial z} & \frac{\partial v_x}{\partial x} + 2\frac{\partial v_y}{\partial y} \\
\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial y} & 2\frac{\partial v_x}{\partial z}
\end{pmatrix}$$

2.5.3. Shallow Ice Approximation

In the shallow ice approximation (Section 5.4 in Greve and Blatter (2009)), further simplifications are made, assuming that in ice sheet ice flow can be represented by simple shear parallel to the bed. This requires the horizontal dimensions of the ice sheet to be much larger than the vertical dimensions, assuming a certain distance to ice divides and ice margins everywhere. This also implies small slopes at surface and bed. Consequently, in the deviatoric stress tensor, the shear components parallel to the bed ($T_{xz}$ and $T_{yz}$) are dominant, and all other components are close to zero: $T_{xy}, T_{yx}$ (shear stresses between vertical planes) and the normal stress deviators $T_{ii}^D$. Due to the relationship

$$T_{ii} = T_{ii}^D - p \quad \quad \quad |T_{ii}^D = 0$$

$$T_{ii} = -p,$$

the Stokes equations of the shallow ice approximation can be expressed as

$$-\frac{\partial p}{\partial x} + \frac{\partial T_{xz}}{\partial z} = 0$$

$$-\frac{\partial p}{\partial y} + \frac{\partial T_{yz}}{\partial z} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g.$$
As already shown in the hydrostatic approximation (Equation 2.9), integrating the third equation yields the expression

\[ p = \rho g (h - z) \]  \hspace{1cm} (2.10)

for pressure \( p \), which can be inserted into the first two equations:

\[ \frac{\partial T_{xz}}{\partial z} = \rho g \frac{\partial (h - z)}{\partial x} = \rho g \frac{\partial h}{\partial x} \]

\[ \frac{\partial T_{yz}}{\partial z} = \rho g \frac{\partial (h - z)}{\partial y} = \rho g \frac{\partial h}{\partial y} \]

Integrating these equations yields

\[ T_{xz} = -\rho g (h - z) \frac{\partial h}{\partial x} \]  \hspace{1cm} (2.11)

\[ T_{yz} = -\rho g (h - z) \frac{\partial h}{\partial y}, \]

which means that the shear stresses between the vertical planes only depend on the thickness of the ice column over a point \( h - z \) and the surface gradient

\[ \nabla h = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix} \]  \hspace{1cm} (2.12)

over this point. As a next step, the stress tensor components \( T_{xz} \) and \( T_{yz} \) will be replaced by the deformation tensor components \( D_{xz} \) and \( D_{yz} \). This is done via Glen’s flow law (Equation 2.7) and will yield equations for the horizontal velocity components \( v_x \) and \( v_y \). Glen’s flow law for \( T_{xz} \) and \( T_{yz} \) is

\[ D_{xz} = A(T') \sigma_e^{n-1} T_{xz} \]

\[ D_{yz} = A(T') \sigma_e^{n-1} T_{yz}, \]  \hspace{1cm} (2.13)

and \( D_{xz} \) and \( D_{yz} \) simplify due to the assumptions made in the first order approximation:
Therefore, Equations 2.13 are transformed into
\[ \frac{\partial v_x}{\partial z} = 2A(T')\sigma_e^{n-1}T_{xz} \]  
\[ \frac{\partial v_y}{\partial z} = 2A(T')\sigma_e^{n-1}T_{yz} \]  

Due to the simplifications of the shallow ice approximation, also \( \sigma_e \) reduces significantly and can be reformulated:
\[ \sigma_e = \rho g (h - z) \left| \nabla h \right| \]  

Inserting this expression in Equations 2.14 gives
\[ \frac{\partial v_x}{\partial z} = 2A(T') (\rho g (h - z))^{n-1} |\nabla h|^{n-1} T_{xz} \]  
\[ \frac{\partial v_y}{\partial z} = 2A(T') (\rho g (h - z))^{n-1} |\nabla h|^{n-1} T_{yz}, \]

and by inserting Equations 2.11, the following equations are obtained:
\[ \frac{\partial v_x}{\partial z} = -2A(T') (\rho g (h - z))^{n} |\nabla h|^{n-1} \frac{\partial h}{\partial x} \]
\[ \frac{\partial v_y}{\partial z} = -2A(T') (\rho g (h - z))^{n} |\nabla h|^{n-1} \frac{\partial h}{\partial y} \]

Integration from the ice basis \( b \) to a ice depth \( z \) yields

\[ v_x(z) - v_x(b) = \int_{b}^{z} -2A(T') (\rho g(h - z'))^{n} |\nabla h|^{n-1} \frac{\partial h}{\partial x} \, dz' \]
\[ v_y(z) - v_y(b) = \int_{b}^{z} -2A(T') (\rho g(h - z'))^{n} |\nabla h|^{n-1} \frac{\partial h}{\partial y} \, dz', \]

where \( v_x(b) = v_{bx} \) and \( v_y(b) = v_{by} \) are the sliding velocities and can be calculated after Weertman’s sliding law (Equation 2.8). Its components can be formulated as follows: \( p_b \) is hydrostatic pressure at the base (Equation 2.10 with \( z = b \)),

\[ p_b = \rho g H \]

and the driving stress \( t_b \) is obtained by setting \( z = b \) in the simplified expression of \( \sigma_e \) (Equation 2.15):

\[ t_b = \rho g H |\nabla h| \]

\( e_t \) is the normalized surface gradient (Equation 2.12):

\[ e_t = \frac{1}{|\nabla h|} \left( \frac{\partial h}{\partial x} \right) \]

Inserting these expressions into Weertman’s sliding law gives

\[ \begin{pmatrix} v_{bx} \\ v_{by} \end{pmatrix} = -C_b \frac{p_b^p}{p_b^q} e_t \]
\[ = -C_b \frac{(\rho g H)^p |\nabla h|^p}{(\rho g H)^q} \cdot \frac{1}{|\nabla h|} \left( \frac{\partial h}{\partial y} \right) \]
\[ = -C_b (\rho g H)^{p-q} |\nabla h|^{p-1} \left( \frac{\partial h}{\partial y} \right) , \]

and substituting the obtained expressions for \( v_{bx} \) and \( v_{by} \) into Equations 2.16 yields
\[
v_x(z) = -C_b (\rho g H)^{p-q} |\nabla h|^{p-1} \frac{\partial h}{\partial x} - \int_b^z 2A(T') (\rho g (h - z'))^n |\nabla h|^{n-1} \frac{\partial h}{\partial x} \, dz'
\]

\[
v_y(z) = -C_b (\rho g H)^{p-q} |\nabla h|^{p-1} \frac{\partial h}{\partial y} - \int_b^z 2A(T') (\rho g (h - z'))^n |\nabla h|^{n-1} \frac{\partial h}{\partial y} \, dz'.
\]

Equations 2.17 change to

\[
v_x = - \left( C_b (\rho g H)^{p-q} |\nabla h|^{p-1} + 2A(T') (\rho g)^n |\nabla h|^{n-1} \int_b^z (h - \bar{z})^n \, d\bar{z} \right) \frac{\partial h}{\partial x}
\]

\[
v_y = - \left( C_b (\rho g H)^{p-q} |\nabla h|^{p-1} + 2A(T') (\rho g)^n |\nabla h|^{n-1} \int_b^z (h - \bar{z})^n \, d\bar{z} \right) \frac{\partial h}{\partial y}.
\]

The integral

\[
\int_b^z (h - \bar{z})^n \, d\bar{z}
\]

can be solved by means of linear substitution via the equation (Merziger)

\[
\int f (ax + b) \, dx = \frac{1}{a} F (ax + b) + C
\]

and its components relate to the integral in Equation 2.18 as follows:

\[
x = \bar{z}
\]

\[
a = -1
\]

\[
b = h
\]

\[
f(\xi) = \xi^n
\]

\[
F(\xi) = \frac{\xi^{n+1}}{n+1}
\]

Therefore,
\[ \int_{b}^{z} (h - \bar{z})^n \, d\bar{z} = -\frac{(h - \bar{z})^{n+1}}{n+1} \bigg|_{\bar{z}=b} \]

\[ = -\frac{(h - z)^{n+1}}{n+1} + \frac{(h - b)^{n+1}}{n+1} \]

\[ = \frac{H^{n+1} - (h - z)^{n+1}}{n+1}. \]

Inserting the solution for the integral gives

\[ \begin{pmatrix} v_x \\ v_y \end{pmatrix} = -C_b (\rho g H)^{p-q} |\nabla h|^{n-1} - 2A(T') (\rho g)^{n} |\nabla h|^{n-1} \frac{H^{n+1} - (h - z)^{n+1}}{n+1} \nabla h, \]

(2.19)

where the term in brackets can be abbreviated as

\[ C = C_b (\rho g H)^{p-q} |\nabla h|^{n-1} - 2A(T') (\rho g)^{n} |\nabla h|^{n-1} \frac{H^{n+1} - (h - z)^{n+1}}{n+1}, \]

leading to an alternative expression of Equation 2.19:

\[ \begin{pmatrix} v_x \\ v_y \end{pmatrix} = -C \nabla h \]

This equation is the final velocity representation of the shallow ice approximation. It can be inserted into \( q \) (Equation 2.3), and the ice thickness equation (Equation 2.4) yields the change in ice height at a certain location \((x, y)\). As the negative sign in front of \( C \) indicates, ice will flow in the opposite direction of the surface gradient.
3. Methods

3.1. Sigma Transformation

In order to maintain the discretisation of the model equations within a certain complexity and hence to maintain computation simple, the aim is to use a rectangular grid with equal grid spacings $\Delta x$, $\Delta y$ and $\Delta z$. The problem in doing so is the highly irregular nature of ice sheets which makes this approach difficult (see left side of Figure 3.1). One strategy to deal with this is to leave the Cartesian coordinate system $(x, y, z)$ and introduce a new coordinate grid $(\xi, \varphi, \varsigma)$ that projects the irregular ice cap onto a rectangular solid (right side of Figure 3.1).

![Figure 3.1. Visualisation of the sigma transform: from an irregular ice cap in the $x$-$y$-$z$ domain to rectangular grid spacings $\Delta \xi$, $\Delta \varphi$ and $\Delta \varsigma$. Figure taken from Greve and Blatter (2009).](image)

The transformation that yields this result is called the sigma transformation and can be calculated as follows:

\[
\begin{align*}
\xi &= x \\
\varphi &= y \\
\varsigma &= \frac{z - b(x, y, t)}{H(x, y, t)} \\
\tau &= t
\end{align*}
\] (3.1)
Here, $x, y$ and $z$ are the Cartesian coordinates which are transformed into $\xi, \varphi$ and $\varsigma$, respectively, while $t$ and $\tau$ are the time before and after the sigma transformation. The basic idea of this transformation is to project the ice surface to $\varsigma = 1$ and the ice base to $\varsigma = 0$, with equidistant grid spacings $\Delta \xi, \Delta \varphi, \Delta \varsigma$ in between.

Following the sigma transformation, not only the coordinates change, but also their derivatives adjust. This is also the case for $\xi, \varphi$ and $\tau$, despite they are defined exactly as $x, y$ and $t$, respectively. After applying the chain rule, these expressions for the derivatives result:

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial \varsigma}{\partial x} \frac{\partial}{\partial \varsigma},
\]
\[
\frac{\partial}{\partial y} = \frac{\partial}{\partial \varphi} + \frac{\partial \varsigma}{\partial y} \frac{\partial}{\partial \varsigma},
\]
\[
\frac{\partial}{\partial z} = \frac{\partial \varsigma}{\partial z} \frac{\partial}{\partial \varsigma},
\]
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \frac{\partial \varsigma}{\partial t} \frac{\partial}{\partial \varsigma}.
\]

In spite of the advantages of the sigma transformation, there are still some problems that are faced when implementing it into the model. First, Equation 3.1 contains a singularity when the ice thickness gets zero. Second, the ice margin will not always be exactly on a grid point, leading to enhanced model uncertainties, particularly in times of fast glacier change.

### 3.2. Discretisation

To solve the system of differential equations numerically, they need to be discretised. Before doing so, a grid has to be defined on which the quantities can be calculated. As mentioned in the previous section, the new coordinates $\xi, \varphi$ and $\varsigma$ on the rectangular domain representing the ice cap will be split by equal grid spacings $\Delta \xi, \Delta \varphi$ and $\Delta \varsigma$. This looks as follows:
$\xi : i = 0, 1, \ldots, I$

$\rightarrow \xi_i = \xi_0 + i\Delta \xi$

$\varphi : j = 0, 1, \ldots, J$

$\rightarrow \varphi_j = \varphi_0 + j\Delta \varphi$

$\varsigma : k = 0, 1, \ldots, K$

$\rightarrow \varsigma_k = \varsigma_0 + k\Delta \varsigma$

Hence, there are $I + 1$, $J + 1$ and $K + 1$ grid points in the directions of $\xi$, $\varphi$ and $\varsigma$, respectively, where one corner of the model domain is defined by $(\xi_0, \varphi_0, \varsigma_0)$. These points are shown as solid circles in Figure 3.2.

Figure 3.2.: 2-dimensional Arakawa C grid. The third dimension $\varphi$ is built in the same manner. Solid circles are part of the main grid, defined by integer-valued $i, j, k$. Open circles denote the staggered grid with half-numbers. Figure taken from Greve and Blatter (2009).

As stated above, $\varsigma = 0$ and $\varsigma = 1$ describe the ice base and surface, respectively. Therefore, $k = 0$ is also the base and $k = K$ denotes the ice surface. $\varsigma_k$ can therefore
also be written as
\[ \varsigma_k = \frac{k}{K}. \]

The grid points at one or more of \( i = 0, i = I, j = 0, j = J \) are the ones closest to the margin. To ensure that the model domain always describes the whole ice cap, the velocity in these cells is set zero to preclude ice flux over the ice margin.

As shown in Figure 3.2, the grid points denoted by integer-valued \( i, j, k \) are not the only grid points that are used to discretise the model equations. In between the main grid lies a sub-grid whose grid points are exactly in the middle between two grid points of the main grid, leading to half numbered values \( i + \frac{1}{2}, i + \frac{3}{2}, \ldots \). The main grid and sub-grid together form the Arakawa C grid, a special form of a staggered grid.

The velocity components are calculated on the sub-grid to relate them directly to the ice fluxes between two main grid cells. On the latter, the other model quantities such as stress tensor \( T \), viscosity \( \eta \) and ice thickness \( H \) are determined. If quantities that are calculated either on the main or on the sub-grid are needed on the other grid points, their values are approximated by interpolation of the values on the neighbouring grid points.

The time \( \tau \) in the sigma transformed model domain is, equivalently to the spatial coordinates, distributed into \( N + 1 \) equally spaced points in time:
\[ t : n = 0, 1, \ldots, N \]
\[ \rightarrow \tau^n = \tau^0 + n\Delta\tau \]

### 3.3. One-step Methods

It is often difficult or even impossible to find analytical solutions for differential equations. In this case, the modeller depends on numerical solutions, at the cost of receiving a solution only on a discrete grid of points. One-step methods have found wide application in diverse science branches. In contrast to multi-step methods, the assigned value of a quantity on a grid point only depends on the last, but not on previously evaluated grid cells. The following paragraphs give a brief overview of some one-step methods.

Since derivatives are the essential parts of differential equations, the central task is to find ways to find substituting expressions for them. A derivative can be simply written as
\[ f(t, u) = u'(t), \]
where \( u \) is a quantity depending on a variable \( t \) and the derivative of \( u \) can be expressed as a function \( f \) depending on both \( u \) and \( t \). In the light of a discrete grid, the derivative can be rewritten in different ways assuming a linear evolution of the quantity between two grid points:

\[
\begin{align*}
    f(t_i, u_i) &= u'(t = t_i) \approx \frac{u_{i+1} - u_i}{h} \quad (3.2) \\
    f(t_i, u_i) &= u'(t = t_i) \approx \frac{u_i - u_{i-1}}{h} \quad (3.3) \\
    f(t_i, u_i) &= u'(t = t_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad (3.4)
\end{align*}
\]

Here, \( u_i = u(t_i) \), \( u_{i+1} = u(t_{i+1}) \) and \( u_{i-1} = u(t_{i-1}) \), and \( h \) is the step size between two grid points, i.e. \( h = t_i - t_{i-1} \). The first way of expressing the derivative (Equation 3.2) is called forward difference since \( u_i \), the value of \( u \) at the point where the derivative is calculated, is used together with its successor \( u_{i+1} \). Similarly, Equation 3.3 is called backward difference because, instead of using the successor, the predecessor \( u_{i-1} \) is taken into account. The third option in Equation 3.4 is the central difference that uses both the predecessor and the successor of a grid point.

Based on these three difference schemes, the one-step methods to estimate a value \( u_i \) are constructed following this relationship:

\[
    u_{i+1} = u_i + h ((1 - w)f(t_i, u_i) + wf(t_{i+1}))
\]

Here, \( w \) is a weight parameter that discriminates between the explicit and implicit share of a method. By varying \( w \), different methods can be created, some of which are displayed in Table 3.3. Below, each of them is described.

<table>
<thead>
<tr>
<th>Method</th>
<th>Weight</th>
<th>Expression for ( u_{i+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler explicit</td>
<td>( w = 0 )</td>
<td>( u_{i+1} = u_i + hf(t_i, u_i) )</td>
</tr>
<tr>
<td>Euler implicit</td>
<td>( w = 1 )</td>
<td>( u_{i+1} = u_i + hf(t_{i+1}, u_{i+1}) )</td>
</tr>
<tr>
<td>Crank-Nicolson</td>
<td>( w = 0.5 )</td>
<td>( u_{i+1} = u_i + h \left( \frac{1}{2}f(t_i, u_i) + \frac{1}{2}f(t_{i+1}, u_{i+1}) \right) )</td>
</tr>
<tr>
<td>Over-implicit</td>
<td>( w &gt; 1 ), e.g. ( w = 1.5 )</td>
<td>( u_{i+1} = u_i + h \left( \frac{3}{2}f(t_{i+1}, u_{i+1}) - \frac{1}{2}f(t_i, u_i) \right) )</td>
</tr>
</tbody>
</table>

Table 3.1: Overview of four different one-step methods and their dependence on the weight \( w \).

**Explicit Euler Method**

Setting \( w = 0 \) in Equation 3.5 yields the explicit Euler method. It can be derived from the forward difference:
Explicit methods have the advantage that they are easy to use as only the known quantities \( u_i, t_i \) and \( h \) are necessary to calculate the value \( u_{i+1} \) of the next grid cell. On the other hand, it is only consistent of first order and its usage is therefore only recommendable for numerically benign differential equations.

**Implicit Euler Method**

The implicit Euler method is obtained by assigning \( w = 1 \). Another way to derive it is via the backward difference:

\[
    f(t_i, u_i) = \frac{u_{i+1} - u_i}{h} \quad \text{and} \quad h f(t_i, u_i) = u_{i+1} - u_i + u_i
    \]

\[
    u_{i+1} = u_i + h f(t_i, u_i)
    \]

Substituting \( i \) by \( i + 1 \) gives the expression for the implicit Euler method:

\[
    u_{i+1} = u_{i+1-1} + \frac{1}{2} h f(t_i, u_i)
    \]

While in the explicit Euler method, the new value \( u_{i+1} \) could be obtained by simply inserting the known values on the right side, now in every iteration step a linear equation system needs to be solved. However, it offers great advantages compared to the explicit Euler method: even for big step sizes \( h \) it yields results that are much more accurate, thus reducing the number of grid points that need to be computed, and it is substantially more stable.

**Crank-Nicolson Method**

Inserting \( w = 0.5 \) into Equation 3.5 yields the Crank-Nicolson method:

\[
    u_{i+1} = u_i + h \left( \frac{1}{2} f(t_i, u_i) + \frac{1}{2} f(t_{i+1}, u_{i+1}) \right)
    \]
It can be derived by applying the trapezoidal rule which results in the arithmetic mean between the implicit and explicit Euler methods. It has a consistency order of two and has shown to be stable for a large number of differential equations. As in the implicit Euler method, in every step a linear equation system has to be solved.

**Over-implicit Method**

Over-implicit methods are obtained when choosing \( w > 1 \). In this case, the solution of the implicit Euler method is upscaled, and the excessive part is subtracted in form of the explicit solution. For \( w = 1.5 \), the expression for \( u_{i+1} \) looks as follows:

\[
u_{i+1} = u_i + h \left( \frac{3}{2} f(t_{i+1}, u_{i+1}) - \frac{1}{2} f(t_i, u_i) \right)
\]

The over-implicit scheme was proposed by Hindmarsh (2004) who also showed that for isothermal ice sheets, it is entirely stable. Since isothermality is assumed for the Mocho-Choshuenco ice cap, the over-implicit scheme is therefore an appropriated choice for this application.

### 3.4. Numerical Solution of the Ice Thickness Equation

The set of equations developed in Chapter 2 need to be solved in a discretised manner in order to obtain the temporal variation of an ice sheet and its properties such as ice thickness and velocity distribution. This is accomplished as follows: First, the horizontal velocity components \( v_x \) and \( v_y \) from Equation 2.19 will be calculated. Afterwards, they are inserted into Equation 2.3 to obtain the ice flux \( q \). Finally, \( q \) is used to calculate the ice thickness evolution using Equation 2.4.

#### 3.4.1. Horizontal Velocity

Equation 2.19 consists of two parts: the term \( C \) and the surface gradient \( \nabla h = \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) \).

Since the velocity components are calculated on the sub-grid, the surface gradient is also needed on the sub-grid and can be calculated using the central difference scheme:

\[
\begin{align*}
\frac{\partial h^{n}_{i+\frac{1}{2},j,k}}{\partial x} &= \frac{\partial h^{n}_{i+\frac{1}{2},j,k}}{\partial \xi} = \frac{h^{n}_{i+1,j,k} - h^{n}_{i,j,k}}{\Delta \xi} \\
\frac{\partial h^{n}_{i,j+\frac{1}{2},k}}{\partial y} &= \frac{\partial h^{n}_{i,j+\frac{1}{2},k}}{\partial \varphi} = \frac{h^{n}_{i,j+1,k} - h^{n}_{i,j,k}}{\Delta \varphi}
\end{align*}
\]

\( C \) is also needed on the sub-grid points. Therefore, its constituents need to be averaged, for example the ice thickness \( H \) in between two main grid points can be calculated following
The absolute value of the surface gradient $|\nabla h|$ can be obtained from its components in $x$- and $y$- direction as calculated above.

### 3.4.2. Vertical Velocity Integration

The next step is to obtain the volume flux $q$ through vertical integration of the horizontal velocity components $v_x$ and $v_y$ (cf. Equation 2.3). This is done by the trapezoidal rule, here shown for the volume flux at the two sub-grid points $i + \frac{1}{2}, j$ (for the $\xi$-component) and $i, j + \frac{1}{2}$ (φ-component) at time $t_n$:

$$
q_x = \int_{h_b}^{h} v_x \, dz = \left( \frac{1}{2} v_{x,i + \frac{1}{2}, j, 0} + \sum_{k=1}^{K-1} v_{x,i + \frac{1}{2}, j, k} + \frac{1}{2} v_{x,i + \frac{1}{2}, j, K} \right) \Delta \xi
$$

$$
q_y = \int_{h_b}^{h} v_y \, dz = \left( \frac{1}{2} v_{y,i, j + \frac{1}{2}, 0} + \sum_{k=1}^{K-1} v_{y,i, j + \frac{1}{2}, k} + \frac{1}{2} v_{y,i, j + \frac{1}{2}, K} \right) \Delta \phi
$$

The velocities in a certain depth $\zeta_k$ can be calculated following the Gaussian quadrature.

### 3.4.3. Ice Thickness Equation

With a discretised $q$, the ice thickness equation 2.4 can be solved. The divergence of $q$ can be rewritten as

$$
\nabla \cdot q = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}.
$$

As both $q_x$ and $q_y$ are known on the sub-grid around the main grid points on which $\frac{\partial H}{\partial t}$ is searched for, the divergence can be calculated using central differences:

$$
(\nabla \cdot q)_{i,j}^n = \frac{q_{x,i + \frac{1}{2}, j}^n - q_{x,i - \frac{1}{2}, j}^n}{\Delta \xi} + \frac{q_{y,i, j + \frac{1}{2}}^n - q_{y,i, j - \frac{1}{2}}^n}{\Delta \phi}
$$

The time derivative is then calculated using the over-implicit scheme as described in Section 3.3, and the source terms $a_s$ and $a_b$ can be simply added to the equation.
4. Study Object

4.1. The Mocho-Choshuenco Volcanic Complex

Until the last glacial maximum (LGM), the Patagonian Andes were covered by the Patagonian ice sheet (Glasser et al., 2008). Deglaciation started around 18000 years ago (Bendle et al., 2019), and since then ice sheet has shrunk to the northern and southern Patagonian ice fields and several smaller ice caps and glaciers (Glasser et al., 2008). Some of these ice caps are located on volcanoes in the Chilean Lake District (Rivera et al., 2006), among them the ice cap covering the Mocho-Choshuenco volcanic complex (Figure 4.1).

![Figure 4.1.: Location of the Mocho-Choshuenco Volcanic Complex in a) Southern South America and b) in the Chilean Lake District.](image)

These volcanoes form part of the active southern volcano zone (SVZ) that experiences volcanic eruptions with a high frequency and ranges 1400 km along the Chilean Andes, starting from the volcanoes near the Santiago in central Chile and reaching towards volcanoes deep into Chilean Patagonia (Dzierma and Wehrmann, 2012). Since the onset of deglaciation, the Mocho-Choshuenco volcanic complex has been among the most active volcanoes in Chile (Rawson et al., 2016). For the last two millennia, a frequency of one explosive eruption every approximately 150 years has been estimated, with the last eruption dating back to 1864 (Rawson et al., 2015).
4.2. Input Data from Glaciological Studies on the Ice Cap

The ice cap covering the Mocho-Choshuenco volcanic complex has shrunken significantly during the past decades: after 1976, when it covered an area of 28.4 km², it lost ice mass with a yearly rate of around 0.4 km² yr⁻¹, leading to a strongly reduced area of 16.9 km² in 2003 (Rivera et al., 2005). Figure 4.2 shows a satellite image of the ice cap in 2015, together with the position of the two summits and an outline showing the ice margins. This outline is also used in the following maps and in the results chapter when comparing the simulated ice cap to the observed one. Due to its good accessibility and infrastructure, many scientific studies have been conducted on the ice cap. The resulting data of some of these studies are used to calibrate the model in this thesis. Therefore, they will be outlined in the following.

Figure 4.2.: Overview map of Mocho-Choshuenco ice cap. Background: Landsat image (February 22, 2015).
Since 2003, surface mass balance (SMB) measurements have been conducted on a stake network on the south-eastern part of the ice cap (red triangles in Figure 4.2) and a negative yearly mean mass balance of $-0.9$ mw e was obtained (Schaefer et al., 2017). In the same study, SMB was modelled based on the measured SMB data and the weather data collected by an automatic weather station (AWS) located near the glacier (AWS-Mocho1 in Figure 4.2). On several of the mass balance stakes, additional GPS measurements were made in 2013 to infer surface velocity of the ice (DGA, 2013). These stakes are marked with an additional black circle in Figure 4.2. The yellow profile connecting Mocho summit with three stakes (B12, B14, B18) shows a flowline that will be used to analyse the model outputs in the results section.

Figure 4.3.: (a) Map of the profiles on which radar data were obtained together with the interpolated ice thickness. (b) Topographic map of the ice base that will be used in this study to simulate ice flow, shown in 100 m horizontal resolution in which most simulations were made. It was obtained by subtracting the interpolated ice thickness data in (a) from a DEM.

With the goal of obtaining information on ice thickness, ground penetrating radar (GPR) measurements were conducted on the ice cap (DGA, 2014). The profiles along which these measurements were made are shown in Figure 4.3a. Interpolation leads to an ice thickness map (Figure 4.3a) and a total volume of 1.038 km$^2$ was inferred (DGA, 2014). After subtracting the ice thickness grid from the elevation data of a digital elevation model (DEM), a topography map of the underlying bedrock was obtained by Flandez (2017), shown in Figure 4.3b. The bed topography is used in this study as the base of the simulated ice cap.
In order to show the locations of some landmarks discussed in the results chapter, two photos taken on the ice cap are shown in Figure 4.4 (location shown in Figure 4.2). Figure 4.4a shows a view of the plateau, together with Mocho summit in the background with its steep slope, and a strong slope break towards the plateau. The ice flows downslope towards the left. The crevasses in the left indicate the beginning of the slope that leads to stake B12. The position of this stake is shown in Figure 4.4b, which is taken from the other side. Consequently, the ice is flowing towards the right here, until meeting a strong slope break under B12 and creating an icefall towards an underlying valley. The position of the first photo (Figure 4.4a) is shown on the left, in vicinity of stake B10 below the slopes of Monte Hess.

Figure 4.4.: (a) Photo taken near stake B10 (position and direction marked by P1 in Figure 4.2) showing Mocho summit on the right, the beginning of the slope towards B12 in the left and in between the plateau of the ice cap. (b) Photo taken between B13 and B12 (P2 in Figure 4.2) showing the slope towards B12, and the valley under the icefall near B12. The peak is Monte Hess, and the position of photo P1 and stake B10 is to its feet on the left. Photos taken by M. Schaefer on May 3, 2018.
5. Results

5.1. Aspect-dependent SMB Parametrisation

The previous results by Flandez (2017) pointed out a basic problem in the simulation of the current ice cap behaviour: The SMB (surface mass balance) parametrisation still does not represent well enough the atmosphere-ice interactions taking place on the glacier. On the one hand, strong winds coming mainly from the north-west redistribute freshly accumulated snow from the exposed slopes to the more sheltered side on the south-east of the summit. On the other hand, the southern slope is more sheltered against incoming solar radiation than the northern part. Mainly due to these two processes, the ELA (equilibrium line altitude is higher in the northern to north-western part than on the slopes south to south-east of the summit.

Taking into account these observations, the first idea to improve the model performance is to vary the ELA of a grid point depending on its relative position to Mocho summit, independent of its distance to the summit. A sinusoidal shape corresponds well to the expected behaviour, and due to its symmetry characteristics, the cosine of aspect will be used to project different slopes to a higher or lower ELA. In section 5.1.1, the equations of this new parametrisation are derived, and in section 5.1.2, it is shown how the resulting simulated ice cap depends on the new parameters.

5.1.1. Derivation of the Parametrisation

The principal idea of this parametrisation is shown in Figure 5.1. With the Mocho summit in the center, ELA should vary depending on aspect $\varphi$ as follows:

- in two opposite, ELA takes an average value $B_{ELA}$
- in the direction $\varphi_0$, ELA takes a higher value $B_{ELA} + A_{ELA}$
- in the opposite direction to $\varphi_0$, ELA is lower, $B_{ELA} - A_{ELA}$

32
Figure 5.1.: Derivation of the new mass balance parametrisation. ELA should take a minimum and maximum on two opposite directions ($B_{ELA} \pm A_{ELA}$), and the mean between them ($B_{ELA}$) on the other directions. Furthermore, $\varphi_0$ serves as direction offset to rotate the values accordingly. ($x_{sum}, y_{sum}$) indicates the position of Mocho summit.

These values can be summarized in a cosine function in $\varphi$ with offset $\varphi_0$ with an amplitude $A_{ELA}$ and $B_{ELA}$, the average ELA:

$$ELA = A_{ELA} \cos (\varphi - \varphi_0) + B_{ELA}$$

(5.1)

Consequently, the cosine takes its maximum at $\varphi = \varphi_0$, and due to the symmetry of the cosine it descends in both directions to the point $\varphi_0 \pm 180^\circ$ on the opposite direction of the summit. $B_{ELA}$ is used to shift the ELA to the desired mean altitude. In the example in Figure 5.1, $\varphi_0$ was set to $30^\circ$.

$\varphi$ is the cardinal direction of a point with respect to the summit and can be calculated by

$$\varphi = \arctan \frac{y}{x} + 90^\circ,$$

(5.2)

where $x$ and $y$ are the distances in the two directions from a grid point to the summit. Since the arctan defines the direction with respect to the positive x axis (eastern direction), a shift by $90^\circ$ is necessary to project it to the positive y axis (northern direction). However, the expression only holds if the summit is on the coordinates $(x, y) = (0, 0)$. 33
To include a summit location \((x_{\text{sum}}, y_{\text{sum}})\) different from the origin of coordinates (see Figure 5.1), Equation 5.2 can be adapted to

\[
\varphi = \arctan \frac{y - y_{\text{sum}}}{x - x_{\text{sum}}} + 90^\circ.
\]

Inserting this into Equation 5.1 gives the final representation to compute ELA based on the set of parameters:

\[
\text{ELA} = A_{\text{ELA}} \cos \left( \arctan \frac{y - y_{\text{sum}}}{x - x_{\text{sum}}} - \varphi_0 + 90^\circ \right) + B_{\text{ELA}}
\]

In the following, this parametrisation is applied with realistic values to the model in order to evaluate its influence on the resulting ice cap behaviour.

Figure 5.2.: (a) Ice thickness distribution for \(A_{\text{ELA}} = 100\, \text{m}, B_{\text{ELA}} = 2050\, \text{m}, \varphi_0 = 315^\circ\). Observed ice cap outline is shown in black and the Mocho and Choshuenco summits are marked as red triangles and the SMB stakes as red crosses. (b) ELA resulting from the parametrisation for the cardinal directions as seen from the Mocho summit.

### 5.1.2. Influence of the Parametrisation

Concluding from observations of mean wind direction and solar radiation distribution on the Mocho-Choshuenco ice cap, it becomes clear that the ELA should be in general higher in the north-west than in the south-west. While the north-western part
is highly exposed to enhanced melt due to short-wave radiation and wind erosion, the south-eastern part is sheltered against both effects by the two summits lying in between. Therefore, setting $\varphi_0 = 315^\circ$ seems reasonable, while previous simulation results suggest an ELA difference of around 150 to 200 m between both sides. Accordingly, $A_{\text{ELA}}$ was first set to 100 m, with $B_{\text{ELA}}$ set to 2050 m.

Figure 5.2a shows the ice thickness distribution for this parameter choice in the upper panel, whereas in Figure 5.2b the ELA parametrisation is visualised. The latter shows the ELA value depending on the cardinal direction to which a slope is oriented, in relation to the Mocho summit in the middle of the ice cap. Here, ELA is highest in the north-west (2150 m), leading to a more negative SMB, and lowest in the south-east (1950 m), resulting in a more positive SMB. In the north-east and south-west, ELA takes its average value $B_{\text{ELA}}$, and in between it follows a sine curve.

The resulting simulated ice cap coincides much more with the observed ice extents than previous simulations. Now, the ice terminus is almost everywhere near the ice limit as seen on satellite images. In the eastern and a fraction of the western part of the ice cap, the simulated ice mass follows closely the observed outline, with only small deviations. However, at several spots the simulated ice extent differs distinctly from the observed one. At three spots, there is significantly less ice than expected: one in the southern part of the ice cap, two in the north near the small plateau between both summits. But the most striking deviations are observed in two large ice tongues that are not observed in reality, one of them originating at the Choshuenco peak and flowing in a western direction, the other at the south-western end of the ice cap, filling a valley below the glacier terminus.

**Variation of the ELA-Amplitude $A_{\text{ELA}}$**

To further illustrate the behaviour of the new SMB parametrisation, now the ELA amplitude $A_{\text{ELA}}$ will be varied. As the previous simulation with $A_{\text{ELA}} = 100$ m mirrored the mean observed ELA difference of 200 m between the north-western and south-eastern faces of the ice cap, in the following a lower value ($A_{\text{ELA}} = 50$ m) and a higher value ($A_{\text{ELA}} = 200$ m) will be tested. To compare directly the influence of different ELA amplitude values, $B_{\text{ELA}}$ and $\varphi_0$ are kept equal. The ice thickness of both model runs are displayed in Figure 5.3.
In Figure 5.3a, the ice thickness distribution for $A_{ELA} = 50\, m$ is shown. As before, the lower diagram shows the resulting ELA for different cardinal directions (Figure 5.3b). The sine wave is much flatter than in Figure 5.2b, leading to an almost constant ELA with little variation with different directions. Consequently, the result resembles the ice extent obtained with constant ELA (Flandez, 2017): in the south, the lower slopes of the observed ice cap are ice-free in the simulation, while in the northern part, more ice than observed is introduced by the model. Apart from the ice tongue originating from the Choshuenco peak that is also visible with $A_{ELA} = 100\, m$ (Figure 5.2a), the outlines of the observed ice cap are reproduced closer and the two gaps that were observed in the previous simulation are filled with ice now.

Figure 5.3c shows the ice thickness distribution and in Figure 5.3d the ELA variation for $A_{ELA} = 200\, m$ is displayed. ELA varies much more than in previous simulations, leading to a difference of 400 $m$ between the north-western and south-eastern parts of the ice cap. As ELA is much higher now in the north, most of the ice to the north of Mocho peak vanishes, leaving only small patches of ice in the vicinity of Choshuenco peak. Again, the ice tongue originating at this peak and flowing down its steep slopes is visible.

On the other hand, the SMB towards the south of the ice cap is higher now than in the
two previous simulations ($A_{\text{ELA}} = 100 \text{ m}$ and $A_{\text{ELA}} = 50 \text{ m}$), leading to more simulated ice here. It flows across the margin of the simulated ice cap, also filling up the previously ice-free tongue directly in the south. To the east of this, the ice tongue originating below stake B12 is again visible, this time even longer and wider than before.

**Variation of Aspect Offset $\varphi_0$**

To evaluate the performance of the aspect offset parameter $\varphi_0$, it will now be varied towards either side of the previously assumed $\varphi_0 = 315^\circ$, testing a maximum ELA in the north ($\varphi_0 = 0^\circ$) and in the west ($\varphi_0 = 270^\circ$). The other parameters are kept the same as in the first simulation ($A_{\text{ELA}} = 100 \text{ m}$, $B_{\text{ELA}} = 2050 \text{ m}$).

As can be seen in Figure 5.4b and Figure 5.4d, the height of the sine waves is now the same, but their offset varies. In the simulation shown on the left, ELA is assumed to be highest in the west (2150 m) and lowest in the east (1950 m). As Figure 5.4a shows, most of the ice mass is shifted towards the north-east, here exceeding the boundaries of the observed ice cap extent east of Choshuenco peak. On the other hand, large parts of the south-east and south are now ice-free, including the valley previously filled by an
ice tongue.

Figure 5.4c displays the ice thickness distribution for a maximum ELA in the northern part of the ice cap. Due to a lower ELA in the south-west, the ice mass is now extending towards lower slopes on this side and beyond the outline of observed ice extent. In the north-east, on the other hand, less ice is visible in this parametrisation.

In conclusion, these results show the varying ice distribution due to the SMB parametrisation. Similar results might be obtained for other values of $\varphi_0$, but the ones shown here resemble best the approximate direction of highest ELA observed on the ice cap.

5.1.3. Steady-State Ice Volume

Figure 5.5 shows the volume evolution of the five discussed model runs. After 500 years of simulation, all models have reached a steady state, i.e. the dynamic behaviour of the glacier stays the same as no parameters are changed during the course of the simulation. The black line indicates the observed ice volume of $1.038 \text{ km}^3$ (DGA, 2014). Some of the simulations supersede the observed ice volume, while one simulated volume is very close to the observed one and one stays below.

![Figure 5.5: Ice volume evolution into a steady state of the five simulations described above.](image)

The initial model run with $A_{\text{ELA}} = 100 \text{ m}$ and $\varphi_0 = 315^\circ$ reaches a volume of around $1.4 \text{ km}^3$ in its steady state (blue line). Even though the ice extent of this simulation
fitted best with the observed outline, the ice volume is around 40% higher than the observed volume.

The ice volume generated by the two model runs with a changed ELA amplitude as compared to the initial run give, diverge strongly from each other. While an ELA amplitude of 50 m yields 0.94 km$^3$ (shown in red), a volume that is slightly smaller than the observed one, an ELA amplitude of 200 m creates an ice cap with a high volume of 1.72 km$^3$ (yellow line).

For the difference in aspect offset $\phi_0$, there is also a difference, but not as high as the previous one: the volume of an ice cap with higher ELA in the west than in the east (purple line) is very close to the observed one, and for a maximum ELA in the north, the volume is slightly higher than for the first simulation (1.45 km$^3$).

5.2. Relevance of Sliding

In Weertman’s sliding law (Equation 2.8), the parameter $C_b$ and the pair of exponents $(p, q)$ can be varied to yield a sliding parametrisation suitable for the respective glacier that is to be modelled. The choice of $(p, q)$ depends mainly on the kind of the glacier bed, and $C_b$ can be used as a calibration parameter to obtain realistic values. Generally, it can be expected that including sliding into the model calculations, i.e. setting $C_b > 0$, will lead to an increased velocity on the glacier (Equation 2.8). With higher velocity, in turn, more mass will be transported to the lower slopes of the glacier (Equation 2.4), leading to a more dispersed, thinner ice mass. The higher amount of ice flowing towards lower elevations will melt away more easily than under non-sliding conditions. This depends on the SMB parametrisation, and therefore sliding not necessarily leads to a wider extension of ice. In the following, sliding experiments with $(p, q) = (1, 0)$ and $C_b = 10^{-4} \frac{m}{atm}$ as well as $C_b = 0 \frac{m}{atm}$ are made to compare the performance of SICOPOLIS on the Mocho-Choshuenco ice cap with and without sliding.

5.2.1. Ice Thickness Reduction

In Figure 5.6, the ice thickness for a simulation run with sliding is compared to one without sliding. The upper two images show the ice thickness distribution with (Figure 5.6a) and without sliding (Figure 5.6b). The general shape is the almost the same in both images. Only at some margins, for example in the south-west, there is less ice with an included sliding parametrisation. Furthermore, the thicker parts of the ice cap (green to yellow colours) are less pronounced in the image with $C_b > 0$. 

39
Figure 5.6.: Comparison between the simulated ice thickness when (a) including sliding and (b) without sliding. Panels (c) and (d) show the respective thickness difference compared to the measured and interpolated ice thickness. Ice thickness difference of less than 10 m is shown in white. The observed ice cap outline is shown in black, the summits in red triangles and the stakes with velocity measurements as red crosses.

In the lower two graphics, the difference between modelled and measured ice thickness distributions is shown. Again, the left panel shows the parametrisation with sliding included (Figure 5.6c), and Figure 5.6d shows the ice thickness difference without sliding. Here, both parametrisations differ strongly from each other, as the whole ice cap is thinner with sliding included. Without sliding, SICOPOLIS overestimates the ice thickness on nearly all the ice cap (Figure 5.6d), whereas the inclusion of sliding leads to a thinner ice mass (Figure 5.6c). Here, only the northern part the ice is thicker than the observations suggest, while on the southern side the modelled ice is thinner than the observed one.

Table 5.1 gives a further insight into the distribution of numerical values within both approaches. It shows that the maximum ice thickness $H_{\text{max}}$ as measured by the radar (258 m) is better approximated by the model run with sliding ($H_{\text{max}} = 264$ m) than for the one without sliding ($H_{\text{max}} = 291$ m). This observation holds also for the mean value.
\( H_{\text{mean}} \) which is reduced by 18 m after including sliding, and therefore coincides better with the real value of 68 m than the value that results by neglecting sliding effects (96 m). Furthermore, sliding reduces the maximum deviation from measured ice thickness by 12% (from 187 m to 165 m) and the mean deviation from 44 m to 24 m (46% decrease). These values underscore the observations that were made when analysing Figure 5.6.

<table>
<thead>
<tr>
<th></th>
<th>( H_{\text{max}} )</th>
<th>( H_{\text{mean}} )</th>
<th>( \Delta H_{\text{max}} )</th>
<th>( \Delta H_{\text{mean}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sliding</td>
<td>264</td>
<td>78</td>
<td>165</td>
<td>24</td>
</tr>
<tr>
<td>No Sliding</td>
<td>291</td>
<td>96</td>
<td>187</td>
<td>44</td>
</tr>
<tr>
<td>Radar</td>
<td>258</td>
<td>68</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1.: Comparison of characteristic ice thickness values between different approaches: A model run with sliding and one without sliding are compared against interpolated radar measurements. The first two columns show the maximum and mean thickness values, and the last two columns indicate maximum and mean values for the difference between modelled and observed ice thickness. All values are in m.

### 5.2.2. Surface Velocity Contrasts

In order to analyse surface velocity on the ice cap, Figure 5.7 shows its distribution over the simulated ice mass. The simulation results for the model run with sliding taken into account are shown in Figure 5.7a, and they are compared to the results without sliding in Figure 5.7b. The general distribution of velocity variation is similar in both cases: especially between both summits and in the south-eastern and south-western margins, ice flows with low velocities of less than 30 m a. On the steeper parts however, as around the summits and towards the margins of the ice cap, velocity reaches higher values of much more than 60 m a. The highest velocities are reached in the northern and southern ice tongues which in reality do not exist.

Although the overall distribution in both simulations is quite similar, some strong differences can be observed in Figure 5.7. Here, velocities that are higher for the sliding model run than for the non-sliding model run are positive and negative values indicate lower velocities when sliding is included. Towards the ice margins, surface velocities are generally lower when sliding effects are included. This behaviour can be observed on several outlet tongues of the ice cap, but especially in the south near the ice tongue where velocity is reduced on a large portion of the area around the stakes.
Figure 5.7.: Absolute surface velocity for the two simulation runs (a) with sliding and (b) without sliding. Panel (c) shows the difference between both velocity distributions, showing the change of including sliding as compared to the non-sliding parametrisation. Velocity differences lower than $3 \, \text{m a}^{-1}$ are displayed in white. The observed ice cap outline is shown in black, the summits in red triangles and the stakes with velocity measurements as red crosses.

The differences between both approaches are also shown by Table 5.2. Here, the velocities as measured at seven stakes in the south-eastern basin of the ice cap (shown as red crosses in Figure 5.7) are compared to the simulation results of both parametrisations. To calculate the latter, absolute velocities on the nine closest grid points to the stake location were averaged.

The observed values range from around $13 \, \text{m a}^{-1}$ at stake B10 which is close to the ice divide above the slopes of Monte Hess to $60 \, \text{m a}^{-1}$ at stake B12 near the ice fall on the western outlet tongue. The values for the stakes on the plateau (B14, B17, B18) vary around $30 \, \text{m a}^{-1}$, and the stakes B8 and B15 flow towards the eastern outlet tongue with velocities of around $20 \, \text{m a}^{-1}$.

In general, the modelled velocities are higher than the observed ones, with magnitude
varying between 0.04 m\textsuperscript{a} and 38.89 m\textsuperscript{a}. An exception is stake B8, where the model underestimates velocity by around 9 m\textsuperscript{a} for both parametrisations. The stakes B14, B17 and B18, which are all located on the plateau, show relatively low deviations between measured and observed values of less than 5 m\textsuperscript{a}. For the stakes near the eastern ice tongue (B8 and B15), absolute deviations between 8 m\textsuperscript{a} and 15 m\textsuperscript{a} are obtained, but velocities at stakes B10 and B12 are strongly overestimated by both parametrisations. This is the case especially for the model run without sliding where velocities at these stakes are around 38 m\textsuperscript{a} higher than the observations indicate.

To contrast the different model runs further, the lowest two lines of the table show mean values and standard deviations obtained by the values of all seven stakes. The observed mean velocity is around 30 m\textsuperscript{a}, which is exceeded by 5 m\textsuperscript{a} and 13 m\textsuperscript{a}, respectively, by the model runs with and without sliding. The standard deviation is also overestimated by both parametrisations, but very close for the sliding model run. While the model run with sliding gives 18 m\textsuperscript{a} which is close to the observed standard deviation of 15 m\textsuperscript{a}, setting the sliding parameter to zero leads to a standard deviation of 27 m\textsuperscript{a}. The behaviour appears mostly due to the highly overestimated values at stakes B10 and B12.

<table>
<thead>
<tr>
<th>Stake</th>
<th>$v_{\text{obs}}$</th>
<th>$v_s$</th>
<th>$v_{\text{ns}}$</th>
<th>$v_s - v_{\text{obs}}$</th>
<th>$v_{\text{ns}} - v_{\text{obs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B8</td>
<td>22.23</td>
<td>12.79</td>
<td>13.97</td>
<td>-9.44</td>
<td>-8.26</td>
</tr>
<tr>
<td>B10</td>
<td>12.67</td>
<td>38.47</td>
<td>51.55</td>
<td>25.80</td>
<td>38.89</td>
</tr>
<tr>
<td>B12</td>
<td>60.30</td>
<td>72.05</td>
<td>98.66</td>
<td>11.75</td>
<td>38.36</td>
</tr>
<tr>
<td>B14</td>
<td>33.76</td>
<td>29.30</td>
<td>37.13</td>
<td>-4.46</td>
<td>3.37</td>
</tr>
<tr>
<td>B15</td>
<td>19.42</td>
<td>28.03</td>
<td>33.87</td>
<td>8.61</td>
<td>14.45</td>
</tr>
<tr>
<td>B17</td>
<td>31.21</td>
<td>33.96</td>
<td>34.09</td>
<td>2.75</td>
<td>2.88</td>
</tr>
<tr>
<td>B18</td>
<td>27.16</td>
<td>27.19</td>
<td>30.15</td>
<td>0.04</td>
<td>3</td>
</tr>
<tr>
<td>Mean</td>
<td>29.53</td>
<td>34.54</td>
<td>42.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>St. Dev.</td>
<td>15.35</td>
<td>18.35</td>
<td>27.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2.: Surface velocities at different stakes: observed velocities $v_{\text{obs}}$ and modelled velocities with ($v_s$) and without sliding ($v_{\text{ns}}$) together with their mean values and standard deviations. Also shown are differences between modelled and observed velocities. All velocities in m\textsuperscript{a}.

5.2.3. Ice Volume Contrasts

Another important parameter to estimate the performance of the model runs is the ice volume. The real ice cap has a volume of around 1 km\textsuperscript{3} and to reach a similar modelled volume is important, since especially future simulations require a good performance to yield realistic results.

Figure 5.8 compares the ice volumes of the simulations with and without sliding. After the typical time span to reach a steady state (around 300 years), the ice cap settles down at volumes of 1.7 km\textsuperscript{3} and 1.4 km\textsuperscript{3} for the runs without and with sliding, respectively.
Even though both values still overestimate the true volume, it becomes clear that neglecting sliding yields a better performance in terms of modelled ice volume.

Another observation is that sliding appears to have an impact on the simulation from an early moment on, as the lines of both simulations already separate before simulating 50 years.

Figure 5.8.: Comparison between the ice volume development of the simulated ice cap with sliding and without sliding. Volume as estimated from measurements is shown as a black line.

5.2.4. Variation along a Flowline

In order to further classify for the spatial variation of ice thickness, Figure 5.9 compares the thickness over a flowline of almost 4 km length that starts at the summit, connects the stakes B12, B14 and B18 and reaches into the southern ice tongue that is created by the model but not existing in reality (see Figures 5.6 and 5.7). This flowline is also very close to a radar track on which ice thickness was measured.

In Figure 5.9a, the modelled ice thickness of both parametrisations along the flowline are displayed in blue and red (sliding and no sliding, respectively) together with the ice thickness measured on the radar track (dashed black line). In all three graphs, a similar general development is observed: First, ice thickness drops sharply where the slope of Mocho crater increases. Between 200 and 800 m from the summit, where the slope is highest, ice is very thin, and from 1000 to 2500 m it reaches its maximum, even though a depression at about 2000 m is visible. After 2500 m ice gets thinner again and at 3200 m, at the glacier terminus, it reaches very low values. The results of
both simulations show a strong thickness increase in the ice tongue created by the model.

As the previous analyses in section 5.2.1 indicated, the simulated ice mass is significantly thicker than the reality that is aimed to reproduce. In this figure, the overestimation is strongest at the crater slope: here the radar measurements suggest only around 20 m of ice thickness, whereas the simulated ice thickness has values of around 45 and 50 m for the simulation with and without sliding, respectively. On the first part of the plateau, until around 2000 m distance from the summit, the blue curve (with sliding included) represents quite well the observational data, whereas the ice mass without sliding is 30 m thicker. The depression at 2000 m is only weakly visible in the radar data, but strong in both simulations. Afterwards, the graph of no sliding starts to adjust better to the observations, whereas the sliding ice cap is up to 20 m thinner. When reaching the glacier terminus, both simulated ice masses are too thick again.

In the second chart (Figure 5.9b), the same lines as in the previous one are sketched, but summed to the bed elevation which is shown as a drawn through black line. This figure adds more observations which are not visible in Figure 5.9a. First, the failure of the modelled thickness is not as clear as before, but they rather seem to be in a good coincidence with the observational data. However, it is observed that the curves of modelled ice thickness are much smoother than the observation, an artefact introduced by the lower resolution of the model. Many of the small depressions and hills seen on the ice surface are not reproduced, especially between 1000 and 2000 m distance from the summit.

Figure 5.9c shows the absolute values of surface velocity along the flowline. Again, the modelled curves are shown in blue (sliding) and red (no sliding). Additionally, three triangles are shown which indicate the location and measured velocity for three stakes which are situated on the flowline. The velocity first rises from 0 m/a at the summit to about 50 m/a at the beginning of the plateau (700 m from the summit). Then, it stays constant until stake B14, which is located 2100 m from the summit. At this point, velocity of both simulation runs start to rise again and reach values of over 100 m/a at the glacier terminus.

Both velocity profiles coincide well in their values until reaching stake B14. Here, velocity is maintained lower in the case of sliding, and increasing higher without sliding (around 25 m/a more than without sliding. After the glacier terminus, in the ice tongue created by the model, velocity increases strongly towards values of 200 m/a and 280 m/a (sliding and no sliding, respectively) and then drops again to 50 m/a in both cases.

The velocities as measured at the stakes are in general well reproduced. The modelled velocities at stake B18 and B14 are very close to the observations. After stake B14, however, the velocities of both model runs diverge, whereby the velocity at stake B12 is only overestimated by 12 m/a by the parametrisation with sliding, and much higher when sliding is not included (refer to Table 5.2 for comparison).
Figure 5.9.: Ice cap parameters along a flowline, extending from the summit (A) to the southern ice tongue (B). (a) Ice thickness of both model runs contrasted to observed radar data, (b) surface elevation in m a.s.l. (meter above sea level) of both model runs, the observed ice surface above the bedrock, (c) surface velocity of both model runs and measured velocities at three stakes along the flowline.
5.3. Influence of Resolution

In order to obtain information about the numerical stability of a simulation code and a chosen parameter ensemble, it is necessary to evaluate different spatial and temporal resolutions of it. Usually, the solution will be unstable for low resolutions, that is, resolution changes lead to changes in the solution. When testing higher resolutions, the solution should become stable and not change any more. However, increasing resolution also strongly increases computational effort. Therefore it is desirable to keep resolution as low as possible to account for available computer infrastructure, but at the same time resolution needs to be high enough to yield interpretable results.

In the following, different spatial and temporal resolutions will be compared to draw conclusions on stability of the simulated ice cap. Apart from the resolution that is changed in every simulation, all other parameters are maintained the same as in the first experiment in Section 5.1.2 \((A_{\text{ELA}} = 100 \text{ m}, B_{\text{ELA}} = 2050 \text{ m}, \varphi_0 = 315^\circ)\).

5.3.1. Spatial Resolution Variation

First, the spatial resolution will be changed with equal temporal resolution. In the results discussed before, spatial resolution was always set to 100 m. Now, one lower spatial resolution (200 m) will be tested together with two higher resolutions: 50 m and 25 m. The ice thickness distributions of the four different experiments are shown in Figure 5.10.

Figure 5.10b shows again the same result as in Figure 5.2 with the same observations: in general, the ice extent is well reproduced, with three parts where too few ice is simulated and two ice tongues that are created by the model but not observed in nature (for more details refer back to Section 5.1.2).

In Figure 5.10a (spatial resolution 200 m), several differences in ice thickness are visible as compared to the run with a resolution of 100 m. Especially in the northern part, more ice is lacking in this simulation, with the long ice tongue originating at Choshuenco peak being much shorter. Additionally, a big ice-free gap appears to the south-east of the summit, together with other inaccuracies in the southern part of the ice cap. However, the overall shape of the ice mass is still preserved.

To obtain the simulation result shown in Figure 5.10c, a spatial resolution of 50 m was chosen, doubling the resolution and thus leading to four times more grid points. This leads to an ice thickness distribution almost equal to the one obtained with 100 m resolution. Again, the northern ice tongue experiences a change, increasing its length by almost 1 km. Together with some minor changes around the Choshuenco peak, this is the only significant deviation from the default run (100 m resolution).

Further increasing spatial resolution to 25 m again leads to a significantly increased computation time, but only to small changes in the ice thickness distribution (Figure
5.10d). The most notable alteration is once more in the ice tongue originating from the Choshuenco peak which changes its shape and gets smaller again compared to a lower resolution.

Figure 5.10.: Ice thickness distribution for different spatial resolutions: (a) 200 m, (b) 100 m, (c) 50 m, (d) 25 m. Temporal resolution is 0.1 a for all runs. All simulations outside this section were done with spatial resolution 100 m.

5.3.2. Temporal Resolution Variation

Now, the temporal resolution will be varied in order to also detect possible changes that can result from a too coarse resolution in time. Spatial resolution will be kept stable at 100 m, and starting from a temporal resolution of 0.1 a (which is the same as in all model runs used before), it will be doubled two times, i.e. temporal resolutions of 0.05 a and 0.025 a will be evaluated. The results can be seen in Figure 5.11.
Figure 5.11.: Ice thickness distribution for different temporal resolutions: (a) 0.1 a, (b) 0.05 a, (c) 0.025 a. Spatial resolution is 100 m for all runs. All simulations outside this section were done with temporal resolution 0.1 a.

Figure 5.11a shows the already known ice thickness from Figure 5.2 with its previously discussed details. When increasing temporal resolution to 0.05 a (Figure 5.11a), almost no changes are visible. The biggest alteration compared to a lower resolution is in the ice tongue originating from Choshuenco peak which has increased in length. Otherwise, the ice cap thickness distribution is the same as in the first image.

After further increasing the temporal resolution (0.025 a, Figure 5.11c), again no larger changes are observed. The ice tongue changes its shape and length again, even though this time also this change is only small. On the rest of the ice cap, no changes result from the different simulation runs.

In conclusion, the influence of both spatial temporal resolution on ice extent is almost negligible as no major changes can be observed, but computation time increases strongly with higher resolution.
5.4. Future Evolution of the Ice Cap

5.4.1. Temperature Projections

Now, the next aim is to model the future development of Mocho-Choshuenco ice cap. In the light of the projected global warming, it is interesting to estimate how the ice cap will develop during the 21st century. In the latest IPCC report (Stocker et al., 2013), four different representative concentration pathways (RCPs) as a measure for greenhouse gases in the Earth’s atmosphere were analysed. They are done based on the decade from which anthropogenic carbon dioxide emissions would go back:

- RCP2.6 is the most optimistic scenario, assuming a decline after 2020,
- RCP4.5 projects a decline of emissions after 2040,
- RCP6.5 assumes reduced emissions past 2080, and
- RCP8.5 suggests no reduction at all until 2100.

In the following, the influence of global warming on Mocho-Choshuenco ice cap assuming the two extreme scenarios (RCP2.6 and RCP8.5) is evaluated. For this reason, temperature projections based on these pathways were extracted from the CMIP5 datasets (Taylor et al., 2012) for the grid point on which the glacier is situated. Figure 5.12 shows the projected increase of temperature until the year 2065.

![Temperature Projections](image-url)

Figure 5.12.: Temperature increase for Mocho-Choshuenco ice cap until the year 2065, based on the CMIP5 datasets for the two scenarios RCP2.6 and RCP8.5.

The temperature projection graphs start in the year 2005 which is assumed to be the year of the steady state of the ice cap as produced in the previous sections. Until the year
2020, both projections yield relatively similar temperature rises, and actually the more optimistic scenario RCP2.6 suggests higher temperatures during the decade between 2010 and 2020 than RCP8.5. However, after 2030 both scenarios diverge significantly, and the temperature difference in the RCP8.5 scenario is almost 2°C, while for RCP2.6 it stays almost constant at around 0.7°C.

5.4.2. Glacier Sensitivity to Temperature Change

To project the 21st century temperature rise as described in the previous section to ice dynamics, it is necessary to find a relationship that connects temperature to some characteristic glacier parameter. A good choice for this is the equilibrium line altitude (ELA) which is a reliable indicator for glacier behaviour and also depends strongly on temperature. For the Mocho-Choshuenco ice cap, there are four years (2009-2013) in which both surface mass balance (and thus the ELA) and temperature data at an altitude close to the ELA are available (Schaefer et al., 2017).

![Regression curve showing the relationship between temperature and ELA.](image)

Figure 5.13.: Relationship between temperature and ELA on Mocho-Choshuenco ice cap. The error bars indicate the error as estimated by Schaefer et al. (2017). A linear regression was calculated to predict ELA based on other temperatures.

These four value pairs are shown in Figure 5.13 together with the ELA errors and a regression curve. The linear function obtained by this regression yields an easy way to predict ELA values based on other temperatures and indicates an increase by 84 m for every °C of temperature increase. A value of this magnitude is commonly obtained and will therefore be used in the following.
5.4.3. Volume Reduction

To get a first overview on the impact of temperature increase on the ice cap, Figure 5.14 shows the development of the ice volume for both simulated scenarios, RCP2.6 and RCP8.5. In total, ice volume is reduced from 1.4 km$^3$ to 0.75 km$^3$ and 0.35 km$^3$ for RCP2.6 and RCP8.5, respectively. In other words, 47% of the ice are melted away by 2065 with the most optimistic temperature increase scenario, and a mass loss of 74% is generated by the more pessimistic scenario.

![Ice Volume Graph](image)

Figure 5.14.: Simulated decrease in volume between 2005 and 2065 for the two scenarios RCP2.6 and RCP8.5.

Since the ELA increases linearly with temperature according to the regression (Figure 5.14), and it is directly connected to glacier geometry, the shapes of the curves of temperature increment are reflected by the ice volume reduction. Between 2015 and 2030, the ice volume of the RCP2.6 scenario is lower than that of the RCP8.5 scenario, following the higher temperature for RCP2.6 between 2010 and 2020 (Figure 5.12). After showing around the same ice volume between 2030 and 2035 (a bit less than 1 km$^3$), both scenarios diverge strongly to yield the final values in 2065 as discussed above.

5.4.4. Ice Margin Retreat

A more detailed view on the spatially varying differences when modelling the ice cap under warming atmospheric conditions is given by Figure 5.15. Ice thickness in 2065 after exposing the ice cap to the emission scenario RCP2.6, i.e. the weakest of the assumptions made in the IPCC report is shown in Figure 5.15a, while in Figure 5.15b the ice cap as modelled under the scenario RCP8.5 is displayed. Both scenarios show strong decreases of the ice extent as well as of ice thickness. In general, the ice retreats from
lower elevations to concentrate around the summits, and thins in areas where ice is still present in 2065.

Figure 5.15.: Ice thickness distribution of the ice cap resulting in 2065 after temperature increase according to the two scenarios RCP2.6 (left) and RCP8.5 (right).

In detail, after 60 years of exposure to the RCP2.6 scenario, the ice boundaries would migrate towards the center of the ice cap, now lying nearly entirely on the plateau. Consequently, the southern ice tongue disappears in this scenario as melt in this region is too high to compensate for less ice supply from above. In addition to this, between the two summits the ice is also significantly vanishing. In the simulations of the current behaviours still a wide ice mass had been observed between the two summits. Now, merely a narrow ice line persists in the higher elevations between them. In contrast to the southern ice tongue, where only a small ice patch persists, the northern ice tongue originating from the Choshuenco peak still persists nearly in its full previous shape.

As expected, the higher temperature increment in the RCP8.5 scenario also leads to stronger ice loss. In 2065, the glacier would have retreated even more than in the previously discussed scenario, and the persistent ice is even thinner. Additionally, a gap appears on the south-eastern slope, giving birth to a new nunatak\(^1\) in the area of today’s stake B14. Between the two summits even more ice disappears. The only stable part, unaffected by all temperature and subsequent ELA increments, is the ice tongue originating from the Choshuenco peak and flowing in a western direction. However, even a small patch of the southern ice tongue is still present.

Table 5.3 shows some characteristic values to illustrate this behaviour. The following parameters are shown for both scenarios (RCP2.6 and RCP8.5) and compared to the values obtained by the present-day simulation: the total ice-covered area \(A_{\text{tot}}\), the maximum ice thickness \(H_{\text{max}}\), the mean ice thickness \(H_{\text{mean}}\). Additionally, the ice volume values as discussed in Section 5.4.3 are shown.

\(^1\)a nunatak is ice-free bedrock surrounded by ice, an ‘island’ in between the flowing ice
Table 5.3.: Characteristic ice thickness and area parameters for the two simulations analysed in this section. The %-rows of volume and area indicate loss with respect to the present-day simulation, while the %-rows of maximum and mean thickness show the relation between the 2065 and present values.

As shown and discussed in Figure 5.15, the area decreases strongly in both scenarios. From the 17.8 km² that were simulated for present-day climate conditions, in 2065 there are 12.4 km² left when applying the RCP2.6 scenario, the area of the ice cap is thus decreasing to 69.7% of its previous extent. For the RCP8.5 scenario, the decrease is even stronger, leading to 8.6 km² in 2065 which is less than half (48.3%) of the present-day value.

Another parameter that is a good indicator for the strong ice loss is the ice thickness. In the present-day scenario, the ice is 264 m at its thickest point, but in 2065, it is reduced to 200 m for the RCP2.6 scenario (75.8% of the original thickness) and to 182 m in the case of the RCP8.5 scenario which is 68.9% of the present-day thickest ice column.

The decrease of the mean ice thickness is even more drastic: While in the simulation of the current ice cap behaviour it has an average of 78 m, the RCP2.6 scenario reduces it to 60 m (76.9%), and the RCP8.5 scenario nearly halves the mean ice thickness to 41 m which is 52.6% of today’s value.

5.4.5. Future Behaviour on a Flowline

To further illustrate the changes induced on the ice cap by the two future greenhouse gas emission scenarios RCP2.6 and RCP8.5, in Figure 5.16 ice thickness, surface elevation and surface velocity are shown on the same flowline on the south-eastern face of the ice cap as in Figure 5.9.

Figure 5.16a shows the variation of ice thickness along the flowline. The present-day distribution is shown in a black dashed line and the two scenarios RCP2.6 and RCP8.5 are displayed in green and red, respectively. During the first about 700 m along the profile, the values of all three graphs coincide well. Further away from the summit, the thinning in the future scenarios is clearly visible. From around 1100 to 1800 m the thickness
Figure 5.16.: Ice cap parameters along a flowline, extending from the summit (A) to the southern ice tongue (B). (a) Ice thickness of future projections contrasted to observed radar data, (b) surface elevation in m a.s.l. (meter above sea level) of both projections and the observed ice surface above the bedrock, (c) modelled surface velocity of both future projections compared to modelled present-day velocities (velocities greater than 60 m/a are clipped).
difference between the three curves are nearly constant. Between 1800 and 2400 m the
difference between the present-day thickness and the RCP2.6 scenario is still constant,
while the curve of the RCP8.5 scenario drops to zero. This location is the same as the
newly developed nunatak that was observed in Figure 5.15b. Afterwards there is again
some ice present, but at 2900 and 3200 m, respectively, the end of the glacier is reached
in the two scenarios.

These observations also project to the surface elevation as produced on the profile for
the three model runs (Figure 5.16b). In the first 800 m, i.e. on the slopes of the Mocho
crater, the curves are almost identical, but when reaching the plateau, they begin to
diverge. At 2000 m, a hill in the bedrock topography that now exceeds the ice thickness
is visible in the RCP8.5 scenario, corresponding to the location of the above observed
nunatak. On the other hand, the ice surface elevation of the RCP2.6 scenario is still
above the hilltop. Between 2400 and 2800 m, there is a small depression which is again
filled by ice in the RCP8.5 scenario. Before the pronounced knick at 3300 m where the
real glacier has its terminus, both future glaciers reach their end.

In Figure 5.16c, the velocity distribution along the flowline is shown. Again, in the
vicinity of Mocho summit, the velocities of all three model runs are similar and diverge
afterwards. In relationship to the velocities modelled by the present-day simulation,
the future scenarios indicate a deceleration of the ice cap. Again, the two future sce-
nario curves are parallel until 2000 m, the difference of the present-day velocity increases
steadily until this point and then diverges strongly. Due to the bump at 2000 m, the
RCP8.5 velocity drops to zero, while the ice mass of the RCP8.5 scenario remains at a
constant, but low value of about 10 m until the glacier terminus at 3300 m. The ice flow
of the RCP8.5 scenario after the nunatak is even smaller with less than 5 m.

Table 5.4 further illustrates the behaviour of the velocity on the profile by comparing
their mean values. While in the present-day simulation the mean velocity on the profile
had been 51.2 m a, in 2065 it dropped to 15.5 m a and 9 m a in the RCP2.6 and RCP8.5
scenarios, respectively.

<table>
<thead>
<tr>
<th>v_mean [%]</th>
<th>RCP2.6</th>
<th>RCP8.5</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.5</td>
<td>9</td>
<td>51.2</td>
<td></td>
</tr>
<tr>
<td>30.3</td>
<td>17.7</td>
<td>30.3</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4.: Comparison of mean velocities of the three scenarios discussed in Figure
5.16c.

In general, the results presented here show significant changes in the behaviour of the ice
cap when exposing it to future climate scenarios. The overall ice mass is reduced strongly
when considering volume, area and flowline length. Also, surface velocity decreases as a
consequence of higher future temperatures.
6. Discussion

6.1. Behaviour and Fate of the Ice Cap

From the results presented in this study, several useful and valuable conclusions regarding the Mocho-Choshuenco ice cap can be drawn. These include information about climate forcing and climate-ice interactions and about the dynamic ice flow behaviour of the ice cap, but also predictions about the fate of the ice cap during the ongoing century can be made.

The main aim of the study was to improve the reproduction of the observed ice cap behaviour made with the ice sheet model SICOPOLIS. In previous studies, a major drawback of the model was its inability to correctly represent the extent of the ice cap, which was shown to be impossible with a spatially constant equilibrium line altitude (ELA) throughout the model domain. Consequently, a new ELA parametrisation was developed in order to resolve locally varying ELA in different parts of the ice cap depending on slope orientation: in principal, in the north-west, ELA should be higher than in the south-east, mainly due to solar radiation and wind deposition effects. The new parametrisation proved to be useful as with varying ELA the ice margins could be reproduced well in most parts of the ice cap: while before there was either too much snow in the north-west with an adequate ice extent in the south-east, or too few snow in the south-east with a good representation in the north-west. Now, both requirements can be fulfilled simultaneously (cf. Figure 5.2).

However, there are still parts of the ice cap in which the simulation results do not coincide well with the observations. Several ice tongues that were observed are not reproduced and additional ice is introduced by the model in other parts, both mostly among the eastern and western margins. These are only minor deviations and can possibly be explained by ice flux deviations in the model due to inaccuracies in higher elevations, for example around the summit. This will be discussed in detail later.
Another drawback of the current model set-up lies in the divergence between observed and modelled ice caps in some areas. Two of them are the two ice tongues in the north (to the south and to the east of the Choshuenco peak) that are seen on satellite images.
but appear white in the simulation result, i.e. no ice is modelled here (Figure 6.1c and 6.1d). The reason for this could lie in the newly introduced SMB parametrisation as it takes into account only the effects of Mocho summit, but not local ELA changes introduced by the Choshuenco peak. Here, Choshuenco peak might also have an influence via shading from solar radiation and enhanced wind deposition. Not including these effects leads to a lack of ice between the Mocho and Choshuenco peaks, and consequently to a lack of ice flowing downhill and filling these two ice tongues. Additionally, the elevation especially of the western ice tongue is rather low, with a minimum elevation of 1650 m (Figure 6.1a). This leads to a very negative modelled SMB (-11 m w.e.) and makes it a candidate region for soon glacier retreat. For comparison, in the south-eastern part, modelled SMB is much higher (-5 m w.e.) at a similar elevation. This shows the high SMB gradient between the north-western and south-eastern parts of the ice cap. An overestimation of this difference by the applied parametrisation is possible but can not be verified due to the lack of measurement data in the north-west.

The modelled volume of the model run with preferred settings yielded a total ice volume of about 1.4 km$^3$, thus overestimating the ice volume estimated by DGA (2014) (around 1 km$^3$ by 40\%). This big difference can be explained by various factors. First, the mean modelled ice thickness was overestimated by 10 m (78 m and 68 m for the modelled and measured ice caps, respectively), and with a modelled ice area of 17.8 km$^2$, this gives an additional volume of around 0.18 km$^3$, accounting for almost half of the modelled volume excess. On the other hand, the two ice tongues in the north and in the south are features commonly produced by the different model runs, and they are not taken into account when estimating the observed volume. The northern ice tongue has a volume of roughly 0.09 km$^3$, and the southern tongue adds another 0.04 km$^3$, giving a sum of 0.13 km$^3$. This overestimation has the same magnitude as the volume excess created by a higher modelled mean ice thickness. Therefore, by these two processes, the high volume difference between model and observation can be partly explained.

A second step of improvement achieved of this thesis is resulting from the sliding experiments that were conducted. As the overall assumption of an isothermal ice cap suggests, a certain fraction of the ice movement is generated by sliding of the ice on the base. A first important improvement induced by including sliding is a thickness reduction of the entire ice cap that is not possible to achieve by SMB parameter manipulation alone. The ice cap simulated with sliding is still thicker than observed in nature, but it resembles the observed ice thickness much better than the corresponding simulation without sliding (Figure 5.6). The thinning following sliding can be explained by more ice running downhill and being melted away, leaving less time for new ice from higher slopes to replace it and thus leading to a thinner overall ice mass, while the ice extent stays the same since the same SMB parametrisation is applied. Another indicator for the improvement of the ice cap reproduction by sliding is surface velocity. The modelled velocities at the locations where stakes velocity data are available are more in line with the observations when sliding is included, i.e. they are lower. This is at first sight a contradiction to the fact that including a basal velocity should increase surface velocity.
However, this behaviour gets clearer when considering the above reflections about thinning of the ice cap. Ice velocity increases from base to surface, and when the surface is found closer to a base, surface velocity will be lower when keeping basal velocity constant.

In addition to these steady-state modelling experiments, two dynamic ice flow model runs were conducted to infer the future development of Mocho-Choshuenco ice cap until the year 2065. As forcing conditions the influence of the representative concentration pathways (RCPs) 2.6 and 8.5 were applied, i.e. the most optimistic and pessimistic future predictions for greenhouse gas development made by Stocker et al. (2013). In both cases, a significant decrease in ice volume until 2065 is observed. The ice loss for the RCP2.6 scenario is 47%, but under the pessimistic RCP 8.5 scenario, 74% of the volume of the steady-state present day simulation will melt away. Moreover, temperature projections suggest a further increase beyond the year 2065. Due to the assumed linear dependence of ELA on temperature (Figure 5.13), ELA will most likely also increase. Consequently, SMB will get even more negative and until the end of the current century it might be high enough to inhibit any further existence of the ice cap.

A strongly negative mass balance during the 21st century has also been predicted for the Northern Patagonia Icefield (Schaefer et al., 2013) which is located towards the south, relatively close to the Mocho-Choshuenco ice cap. Results obtained by numerical ice sheet modelling on San Rafael glacier, located on the northern icefield, come to a similar conclusion (Collao-Barrios et al., 2018). However, these results are difficult to compare directly to the Mocho-Choshuenco ice cap since it is much smaller than the Northern Patagonia Icefield. Therefore, although large ice losses are predicted, they are not in risk of disappearing as the Mocho-Choshuenco ice cap. On the other hand, tropical glaciers in the northern Andes are retreating fastly (Vuille et al., 2018). Although the climatic conditions are very different to southern Chile, it is worth comparing them to the Mocho-Choshuenco ice cap due to their similar size. Réveillet et al. (2015) applied Elmer/Ice to Zongo Glacier in the vicinity of La Paz, Bolivia, and found a volume loss of 40% and 89% until 2100 for the two scenarios RCP2.6 and RCP8.5, respectively. These projected relative ice mass loss estimates are lower than the ones predicted in this thesis for Mocho-Choshuenco ice cap, where until 2065 already 47% and 74% (RCP2.6 and RCP8.5, respectively) ice volume reduction is expected. The percentage of the RCP8.5 scenario can be expected to become greater than 89% in the remaining 35 years of the century. This result emphasizes the high vulnerability of Patagonian glaciers to global warming compared to other climate zones.

6.2. Performance of the Shallow Ice Approximation

Apart from results regarding the ice cap itself, also important conclusions about the shallow ice approximation (SIA) can be made. Together with the work done by Flandez (2017), this was the first attempt to apply SICOPOLIS and the SIA to a very small ice cap. Before, only large ice masses such as the ice sheets had been considered in
the SICOPOLIS community, with the smallest modelled object being the Austfonna ice cap on Svalbard (Dunse et al., 2011), whose area is over 400 times bigger than the one of Mocho-Choshuenco ice cap. Therefore, it can be regarded as a success that the simulations performed here yielded an artificial ice cap that in many aspects resembles the real one. Especially in the plateau region with its wide horizontal ice surface, the SIA yields good results.

Figure 6.2.: Close-up view of the northern ice tongue: a) bed surface elevation, b) bed surface slope, c) modelled surface velocity. The observed ice extent is shown as a black line and Choshuenco peak is displayed as a red triangle.

When comparing different resolutions, the modelled ice cap stays stable everywhere except for the northern ice tongue which is further discussed below. The independence of the results from both spatial and temporal resolutions indicate numerical consistency,
as could be assumed by choosing the over-implicit scheme to solve the discretised model equations.

However, the SIA gets close to its limits in several points of the present simulations. One limitation can be found at the slopes of Mocho summit which on satellite images are largely ice-free, while by most simulations, a connected ice cap all over the summit is created. A possible reason for this can be found in the shallow ice approximation, where the relevance of the stress stresses \( T_{xz} \) and \( T_{yz} \) is increased due to the assumption that ice flows parallel to the horizontal plane. As a consequence, the normal components of the deviatoric stress tensor \( T^D_{xx} \) and \( T^D_{yy} \) are neglected (see Section 2.5.3). These describe longitudinal variations in flow direction which at high slopes, and especially at slope breaks, are not negligible anymore. Consequently, these simplifications lead to errors when simulating areas such as the summit region, where one slope break exists at the end of the crater and the beginning of the steep summit inclination, and another one at the lower part of this slope where the plateau begins.

Another area in the simulation domain where the simulations produce an obvious error is the ice tongue that originates near the Choshuenco peak and flows in a westward direction through a narrow valley (Figure 6.2a) with steep side walls (Figure 6.2b). The exact location of the ice together with its surface velocity is shown in Figure 6.2c. It remains stable throughout all simulations where SMB parametrisation and sliding behaviour are varied, and even persists under the most pessimistic future climate prediction (RCP8.5). When varying spatial and temporal resolutions, the general form stays the same, but its length varies, and it is the only location in the model domain where changes occur under different resolutions. These observations suggest numerical errors to be responsible for the development of the ice tongue.

The equal size of the ice tongue in most simulations suggest, when comparing it to the very different SMBs between the simulations, an infraction of the mass conservation principle which states that mass can be neither created nor destroyed (Section 2.1). In SICOPOLIS, this principle is represented by the ice thickness equation (Equation 2.4). Here, two factors have an influence on temporal variation of ice thickness: SMB and velocity difference. SMB is always constant, but the velocity in this area acquires unrealistic high values (Figure 6.2c). In fact, there are two grid points of very high velocity (800 m and 900 m), surrounded by several points with significantly lower, but still very high velocities (between 200 m and 300 m). It is therefore likely that the numerical representation of the equation system is not able to handle these steep velocity gradients. As a consequence, mass might be created, leading to a violation of mass conservation. However, the terrain in this area is very steep (Figure 6.2a and 6.2b), leading to the same problem as above when the slopes of the Mocho summit were discussed. Another failure of the SIA in this area might be the negligence of the shear stress \( T_{xy} \) which describes lateral stress. The ice tongue flows through a valley with very steep walls (Figure 6.2), and these should have a restricting influence on mass movement. Neglecting \( T_{xy} \) might therefore be partly responsible for the high velocity in the ice tongue. It is difficult to
find the exact reason for the formation of the ice tongue, but possible influences are numerical failure and failure introduced by the SIA.

Even though persisting in many modelling attempts, the southern ice tongue behaves differently than its northern counterpart in several ways. First, most of its ice mass disappears under future climate scenarios, and it is therefore less intractable than the northern ice tongue. It also varies under different SMB parametrisations, for example when applying a maximum ELA in the west (Figure 5.4) it disappears. Additionally, it reaches a high maximum velocity of 400 m a\(^{-1}\) (Figure 6.3c) which is very high, but less than the one observed in the north. This high velocity value is observed below a slope break which can be identified in Figure 6.3a near the outline of the observed ice cap. However, the velocity above the slope break is already very high, with velocities between 100 m a\(^{-1}\) and 200 m a\(^{-1}\).

![Figure 6.3.](image)

**Figure 6.3.** Close-up view of the southern ice tongue: (a) bed surface elevation, (b) bed surface slope, (c) modelled surface velocity. The observed ice extent is shown as a black line and the locations of the stakes are displayed as red crosses.
Above that, a striking observation can be made observing satellite images on Google Earth. As shown in Figure 6.4, there is actually ice in the discussed valley, even though strongly debris-covered. Protected by debris and strongly sheltered by the surrounding steep slopes (Figure 6.3b), it is able to persist in spite of its low elevation (1200 - 1500 m). Consequently, this observation indicates that the error of the model in this area is not as severe as might be expected, since current SMB conditions seem to allow ice to exist here. However, the high modelled velocity of this ice tongue is unrealistic as the ice observed on Google Earth is most probably dead ice and the modelled velocity can be attributed to the SIA due to high slopes and slope breaks. Also, the observed ice mass has a length of about 200 m and a width of 100 m and is therefore much smaller than the modelled ice tongue.

![Figure 6.4.: Ice observed on satellite images (red ellipse) at the same location where the model creates the southern ice tongue discussed in the text. Source: Google Earth (March 2019).](image)

Rivera et al. (2005) identified the ice tongue as part of the ice cap in satellite images in 1976 and 1987, while until 2003 a retreat to elevations above the slope break was observed. A possible explanation is the start of strong debris coverage during this 16-year time period, leading to darker colours on satellite images and hence a classification as an ice-free area. Currently conducted analyses of a series of satellite images of the ice cap spanning over several decades might give further insights into the evolution of this ice tongue.
6.3. Recommendations and Outlook

The results obtained in this study yield valuable conclusions to increase our understanding about the Mocho-Choshuenco ice cap and the shallow ice approximation (SIA). However, there are several ways of further improving both the quality and the quantity of the results.

One important step is to further work on the surface mass balance (SMB) parametrisation. The newly developed parametrisation was able to improve the reproduction of the ice cap significantly, but there are still parts of the ice cap that are not well simulated. This is the case especially in the north, where the influence of the Choshuenco peak on SMB is still not included. It might be a solution to add a further equilibrium line altitude (ELA) manipulation based on position relative to Choshuenco peak. Also, the exact aspect of a grid point could be taken into account to resolve local ELA differences further. However, these attempts have to be made with caution since every further parameter that is introduced into the model increases its complexity. This might be against the overall aim of using models that are as simple as possible.

Additionally, there is still space to improve the future simulations. One idea is to simulate more years of the future to cover the whole 21st century which improves comparability to other work that has mostly been done for this period. Besides, it might be interesting to see the behaviour of the ice cap under further future scenarios such as the RCP4.5 and 6.5 scenarios.

The usage of the SIA on a small ice cap has been successful, but it is necessary to compare the results with other models of higher order that include the full system of Stokes equations. One option would be the open-source model Elmer/Ice that has been applied to a wide range of ice masses, also to smaller glaciers and ice caps, e.g. to San Rafael glacier on the Northern Patagonia Icefield (Collao-Barrios et al., 2018). Results of such a model could confirm or falsify the conclusions drawn here regarding the inability of the SIA to correctly model several parts of the ice cap.

Valuable improvements of the simulation results on the Mocho-Choshuenco ice cap can also be gained by further in-situ measurements. Until now, surface mass balance measurements have only been made on the south-eastern face of the ice cap. To obtain a greater spatial distribution of SMB information and to verify the correctness of the applied SMB parametrisation, installing stakes on the northern part of the ice cap would be very helpful. Surface velocity measurements are also only available on a very sparse set of points on the south-eastern face of the ice cap. Data with a higher spatial and temporal distribution would enable a more detailed analysis of the model outputs.

The radar tracks which were used to reconstruct the elevation of the glacier base cover most of the ice cap (Figure 4.3a), leading to an interpolated ice thickness map of already high quality (Figure 4.3b). However, there are some gaps worth filling, especially
between the Mocho and Choshuenco peaks where no measurements have been made so far and inferred ice thickness depends highly on the applied interpolation method. This region is of particular interest as it is the accumulation area for the two northern ice tongues that are not yet well represented by the model. These ice tongues are also a target worth studying in more detail. Another possibility to improve the bed topography and subsequently the simulation results would be to smooth the interpolated ice thickness map before subtracting it from the DEM to obtain the bed topography. The current, not smoothed topography contains relatively deep local depressions that are a direct result of ice thickness maxima, and these violate the SIA assumption of a flat and smooth bed. Brinkerhoff et al. (2016) state that ice thickness and therefore bed elevation should be constrained by a certain degree of smoothness via a correlation length parameter. They also provide a method of inferring ice thickness without direct thickness measurements via probabilistic inverse modelling which could be used. This or a similar method can be used to improve the simulations of the ice cap in two ways. First, it can possibly be used to fill data gaps such as in the northern part of the ice cap. Apart from that, an independent method can provide an estimate about how accurate the derivation of the ice thickness as applied here is. Both the radar measurements and the interpolation introduce errors into the ice thickness map. In further studies, a sound error analysis on both these influences needs to be made to be able to evaluate the model errors in more detail, for example regarding the overestimation of ice thickness and volume.

Concerning the study object, to go beyond the Mocho-Choshuenco ice cap is one goal for future studies. A next step worth doing would be applying SICOPOLIS to Patagonian glaciers, since these glaciers offer a challenging model environment with high climate and subsequent SMB gradients (Schneider et al., 2003) and a wide range of different dynamic behaviour (Sakakibara and Sugiyama, 2014). The great number of calving glaciers in Patagonia also provide a good study object for the complex topic of ice-water interaction (Truffer and Motyka, 2016). Over the last years, many lake measurements have been performed in Patagonia (Sugiyama et al., 2016), providing useful data for model validation.

As a final conclusion to this thesis, it can be stated that the simulations performed here provide valuable information on both the studied ice cap and the applied ice sheet model. However, these achievements should be seen only as a first step preparing further investigations. These include more detailed analyses of the ice cap, both in the acquisition of new field data and in the processing, modelling of these data and processing the model outputs. This holds not only for the Mocho-Choshuenco ice cap and comparable ice-covered volcanoes in the region, but also for the remote and wild glaciers and ice caps of the Patagonian Andes.
A. Stress and Strain Rate Tensors

A.1. Cauchy’s Stress Tensor

Cauchy’s stress tensor $T$ in three dimensions is defined as follows:

$$T = \begin{pmatrix}
T_{xx} & T_{xy} & T_{xz} \\
T_{yx} & T_{yy} & T_{yz} \\
T_{zx} & T_{zy} & T_{zz}
\end{pmatrix}$$

In $T$, all volume forces acting on a volume element are condensed. The first index shows the axis along which a force acts, and the second index indicates the surface to which the force acts (Gerya, 2009). Therefore, the normal stresses $(T_{xx}, T_{yy}, T_{zz})$ are located on the main diagonal, since in their case the axis along which the force acts stands perpendicular to the surface. A positive normal stress element $T_{ii}$ denotes compressive stress, while a negative normal stress element stands for tension. The off-diagonal elements are the shear stresses $(T_{xy}, T_{yx}, T_{xz}, T_{zx}, T_{yz}, T_{zy})$ where the force acts parallel to the surface. Under angular momentum balance, Cauchy’s stress tensor is symmetric (Greve and Blatter, 2009) and thus the shear stresses acting on a specific surface along a specific axis are the same as with exchanged axes and surface:

$$T_{ij} = T_{ji}$$

Pressure $p$ is defined as the negative mean normal stress (Gerya, 2009):

$$p = -\frac{1}{3}T_{ii} = -\frac{1}{3}(T_{xx} + T_{yy} + T_{zz})$$

It is positive under compression and an invariant, i.e. it does not change when applying another coordinate system. It is used to define the deviatoric stress $T^D$, which shows the deviation from the hydrostatic stress state:

$$T^D_{ij} = T_{ij} + p\delta_{ij},$$

where $\delta_{ij}$ is the Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$
Consequently, the deviatoric stress tensor can be written in matrix form as follows:

\[
T^D = \begin{pmatrix}
T_{xx} + p & T_{xy} & T_{xz} \\
T_{yx} & T_{yy} + p & T_{yz} \\
T_{zx} & T_{zy} & T_{zz} + p
\end{pmatrix}
\]

Summing up the normal deviatoric stresses gives zero,

\[
T_{xx}^D + T_{yy}^D + T_{zz}^D = T_{xx} + p + T_{yy} + p + T_{zz} + p
\]

\[
= T_{xx} + T_{yy} + T_{zz} - 3 \cdot \frac{1}{3} (T_{xx} + T_{yy} + T_{zz}) = 0, \quad (A.1)
\]

and the deviatoric shear stresses stay the same:

\[
T_{ij} = T_{ji} = T_{ij}^D = T_{ji}^D, \quad j \neq i
\]

In addition to pressure which is referred to as the first invariant, another invariant of Cauchy’s stress tensor is the second invariant,

\[
II_T = \frac{1}{2} \text{tr} (T)^2,
\]

which is used in Section 2.3 about Glen’s flow law.

### A.2. Strain Rate Tensor

The strain rate tensor \( D \) relates the velocity gradients in a volume element to deformation and, leaving out translational motion, takes the form

\[
D = \frac{1}{2} \begin{pmatrix}
2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\
\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} & 2 \frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\
\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} & 2 \frac{\partial v_z}{\partial y} + \frac{\partial v_z}{\partial z}
\end{pmatrix},
\]

or in a more compact notation

\[
D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \frac{1}{2} \left( \nabla v + (\nabla v)^T \right). \quad (A.2)
\]

Greve and Blatter (2009) showed that the main diagonal elements \( D_{ii} \) indicate the dilatation rate in the respective directions, and the non-diagonal elements equal half the shear rate. Summing up the dilatation rates yields the divergence of velocity:

\[
D_{ii} = D_{xx} + D_{yy} + D_{zz} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot v
\]
Bibliography


DGA (2013). Implementacion nivel 2 estrategia nacional de glaciares: Mediciones glaciologicas terrestres en chile central, zona sur y patagonia”.


Stocker, T., Qin, D., Plattner, G., Tignor, M., Allen, S., Boschung, J., Nauels, A., Xia, Y., Bex, V., and Midgley, P. (2013). Ippc, 2013: summary for policymakers in climate change 2013: the physical science basis, contribution of working group i to the fifth assessment report of the intergovernmental panel on climate change.


Eidesstattliche Erklärung


Freiberg, 14. März 2019

Matthias Scheiter
Declaration

I hereby declare that I completed this work without any improper help from a third party and without using any aids other than those cited. All ideas derived directly or indirectly from other sources are identified as such. This declaration also refers to the representation of figures and visual material.

Freiberg, March 14, 2019

Matthias Scheiter
Acknowledgements

This thesis is coming to an end and I would like to thank all the people that have helped me to get to this point. First of all, I want to thank Prof. Dr. Klaus Spitzer for his dedication and support towards me from the first moment and for opening many doors for me. I had the opportunity to work under the supervision of Dr. Marius Schaefer for more than three exciting and successful years. During this time, I have learned a lot from him, both in an academic and personal way and I hope that we can continue our collaboration and friendship in the future. Dr. Ralf Greve has introduced me with much effort and time to his brilliant ice flow model. Without him, none of the work done in this thesis would have been possible. Furthermore, I want to give thanks to Eduardo Flandez for sharing his results and experience with me. I feel a deep gratitude towards Birk Härtel for his detailed and demanding comments on my thesis and for our great friendship. I would like to thank all my friends for sharing their time with me and making the last five years a time full of freedom and joy. Most importantly, I thank my family for their unconditional support and the exceptional cohesion between us. Finally, I acknowledge the PROMOS scholarship from DAAD during my time in Chile.